MATH 61-02: WORKSHEET 8 (§5.3-5.4)
AND PRACTICE PROBLEMS FOR MIDTERM 2

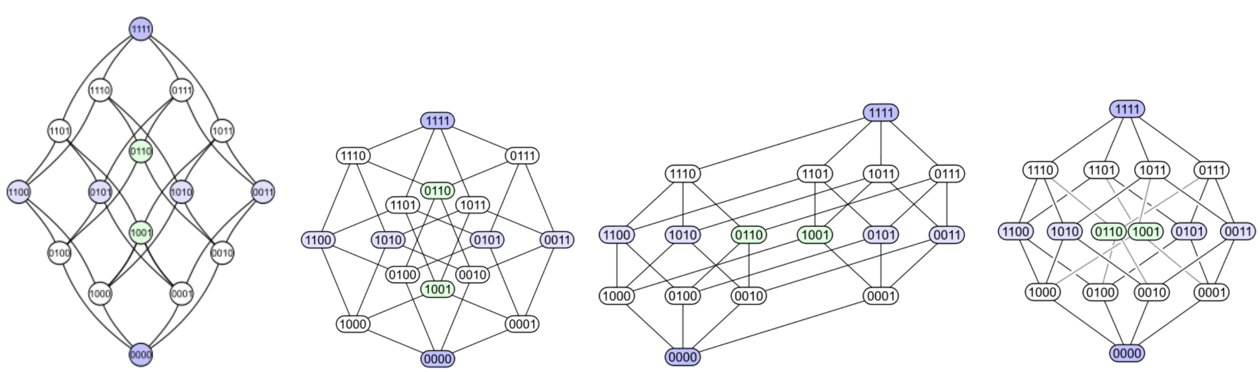
(W1) Let $(P, \leq)$ be a poset. A *chain* is a sequence of distinct elements $x_1 \leq x_2 \leq ... \leq x_k$, and we say that k is the *length* of the chain. An *antichain* is a subset $A \subset P$ such that $x \| y$ for all $x, y \in A$, and we say that $|A|$ is the *width* of the antichain. (In other words, a chain is a subset of P in which any two elements are comparable; an antichain is a subset of P in which no two elements are comparable.)

(a) We have seen that $(\mathcal{P}([n]), \subseteq)$ is a poset. What is the length of the longest chain in this poset?

(b) Recall that $Sub_k(S)$ is the set of all k–element subsets of S. Verify that for any $0 \leq k \leq n$, the poset contains $Sub_k([n])$ as an antichain. What is its width?

(c) Let α be the length of the longest chain, which you computed in the first part. It is a general fact about posets that they can be partitioned into α antichains. Verify that in this case.

(d) The following are four Hasse diagrams of $(\mathcal{P}([4]), \subseteq)$. Which one is organized to make the chains and antichains easy to recognize? Explain.



(W2) We saw several examples of topological quotient spaces in class. For instance, $[0,1]/0 \sim 1$ is a circle.

(a) Consider the equivalence relation on $\mathbb{R}$ given by $x \sim y \iff x - y \in \mathbb{Z}$. What is the equivalence class $[0]$? What is the equivalence class $[\pi]$? Describe $\mathbb{R}/\sim$.

(b) Let $\mathbb{D}$ be the unit disk $\{(x,y) : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$. Define an equivalence relation by

$$(x,y) \sim (z,w) \iff (x,y) = (z,w) \text{ or } x^2 + y^2 = z^2 + w^2 = 1,$$

and describe the resulting quotient space $\mathbb{D}/\sim$.

(c) Come up with an equivalence relation that turns an annulus $\mathbb{A} = \{(x,y) : 1 \leq x^2 + y^2 \leq 4\}$ into a torus.

(PP1) In a lottery, 5 balls are drawn at random from 81 balls numbered 1 through 81.

(a) How many ways are there to draw balls that are all odd-numbered?

(b) What is the probability that the five balls you draw will be consecutive?

(c) How many ways to draw balls such that no two are consecutive? (If this is too hard, try it with 2 balls out of 10 instead of 5 out of 81.)

(PP2) (a) Suppose $|A| = 10$ and $|B| = 8$. How many injections are there from $A \to B$?

(b) For the same A and B, how many surjections are there from $A \to B$?

(c) Give a bijection from the integers $\mathbb{Z}$ to the odd integers $2\mathbb{Z} + 1$.

(PP3) Let L be a line in the plane. Let $\mathcal{S}$ be the set of lines in the plane not parallel to L. Define a relation $\sim$ on $\mathcal{S}$ by

$$L_1 \sim L_2 \iff L_1 \cap L = L_2 \cap L$$

(so two lines are related if they intersect L in the same set).

(a) Show that $\sim$ is an equivalence relation.

(b) Determine the equivalence classes of $\sim$, and describe the quotient space $\mathcal{S}/\sim$.