Math 61-02: Second Midterm: SOLUTIONS

(1) [20 mins] *Consider a set* $S = \{x, y, z\}$. *As you know, a* relation *is any* $R \subseteq S \times S$. *(For instance,* equality *is the relation* $R_0 = \{(x, x), (y, y), (z, z)\}$, *which has three elements.)*

That means that the set of all relations on S *is* $\mathcal{R} = \mathcal{P}(S \times S)$. *Note that the relations themselves form a poset, ordered by* $\subseteq$.

(a) *Does the poset* $(\mathcal{R}, \subseteq)$ *have a minimum and a maximum?*

Yes, $\emptyset$ is the minimum because $\emptyset \subseteq R \quad \forall R \in \mathcal{R}$.
On the other hand, $S \times S$ is the maximum because $R \subseteq S \times S \quad \forall R \in \mathcal{R}$.

(b) *How many different relations are there on* S*?*

Since $|S| = 3$, the Cartesian product has order $|S \times S| = 9$, which means that $|\mathcal{R}| = |\mathcal{P}(S \times S)| = 2^9 = 512$.

(c) *Give an example of a symmetric relation on* S *with five elements.*

$R = \{(x, x), (x, y), (y, x), (x, z), (z, x)\}$ is symmetric because for each pair in the relation, its mirror image is also in the relation.

(d) *Give an example of a non-symmetric relation on* S *with one element.*

$R = \{(x, y)\}$ works, or indeed any relation $R = \{(a, b)\}$ where $a, b \in S$ and $a \neq b$.

(e) *What is the definition of a* function $f : S \to S$*? What is the probability that a random relation on* S *is a function?*

A function $f : S \to S$ is a relation $f \subseteq S \times S$ such that $\forall s \in S$, $\exists! s' \in S$ such that $(s, s') \in f$. In other words, every input from S has exactly one associated output from S.
How many of these are there? Well, for each of the 3 inputs, we must choose one output; there are 3 ways to do that. Since these choices are independent, there are $3 \cdot 3 \cdot 3 = 3^3 = 27$ functions $S \to S$. So since we already computed the number of relations, we find that the probability that a random relation is a function is $9/512$, or just under 2%.

(f) *Briefly verify that* $R = \{(x, x), (x, y), (y, y), (y, x), (z, z)\}$ *is an equivalence relation. What is the cardinality of the quotient space* S/R*?*

Reflexive: we check that the entire diagonal is present. We have $(x, x), (y, y), (z, z) \in R$. ✓
Symmetric: we must check that for each pair in the relation, its mirror image is also present. This is only nontrivial to verify for off-diagonal pairs, which are (x, y) and (y, x). ✓
Transitive: we must check that $(a, b), (b, c) \in R \implies (a, c) \in R$. This is only nontrivial to verify for off-diagonal pairs, which are (x, y) and (y, x). Those chain in both directions, which requires that (x, x) and $(y, y) \in R$. ✓
We have xRy under this relation, which means that $[x] = [y]$, but there is no relation identifying z with either of the other elements. Thus $S/R = \{[x], [z]\}$, which has cardinality two.
(Note the test asked for $|R/\sim|$ rather than $|S/R|$, but this was a typo, so this part of the question was not given a score.)

(g) Prove that if R_1 is a reflexive relation and $R_1 \subseteq R_2$, then R_2 is reflexive as well.

A relation R on S is reflexive iff it contains the diagonal (i.e., $\Delta_S \subseteq R$). But then R_1 reflexive $\implies \Delta_S \subseteq R_1$, and $R_1 \subseteq R_2$, so R_2 contains the diagonal as well.

(2) [16 mins]

(a) *Let $W = \{a, b, \ldots, y, z\}$ be the western alphabet and let $G = \{\alpha, \beta, \ldots, \phi, \omega\}$ be the Greek alphabet; they have 26 and 24 letters, respectively. How many functions $f : W \to G$ are surjections with $|f^{-1}(\omega)| = 3$?*

There must be 3 elements of W that map to ω, so there are $\binom{26}{3}$ ways to select them. There remain 23 elements of W to map to the other 23 elements of G, so there are 23! ways to arrange them. Thus the overall number of surjections is $\binom{26}{3} \cdot 23!$.

(b) *Let g be a function from $\{N, S, E, W\}$ to $\{\heartsuit, \clubsuit, \diamondsuit, \spadesuit\}$. Prove that if g is injective, then g is bijective.*

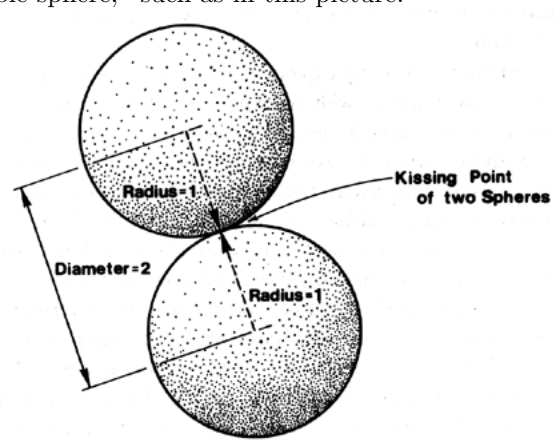
Here, the source and the target each have cardinality four. If g is injective, that means that $|g^{-1}(t)| \leq 1$ for each $t \in f(S)$. For any function at all, the union of the preimages of target elements is the whole source set (because every source element must map somewhere!). The only way to add four numbers ≤ 1 to get 4 is if they are all 1, so each target element has one preimage. That means each target element is hit by some source, so the function is surjective.

(c) *Let $\mathcal{F}$ be the set of all finite sets. Consider the relation Ⓑ on $\mathcal{F}$ that is defined by X Ⓑ Y if and only if there exists a bijection from X to Y. It is clearly reflexive and symmetric. Briefly explain why it is transitive and describe the quotient space $\mathcal{F}/$Ⓑ .*

Let's prove a lemma: there exists a bijection between finite sets X and Y if and only if $|X| = |Y|$. Forward direction: suppose there's a bijection $X \to Y$, and suppose $X = \{x_1, \cdots, x_n\}$. Then I can label the elements of Y by writing $y_i = f(x_i)$, which means $Y = \{y_1, \cdots, y_n\}$, and they have the same cardinality. Backward direction: suppose $X = \{x_1, \cdots, x_n\}$ and $Y = \{y_1, \cdots, y_n\}$. Then I can define a function by sending $x_i \mapsto y_i$, which is clearly injective and surjective.
Now it's clear why Ⓑ is reflexive (every finite set has the same cardinality as itself!) and symmetric (if $|X| = |Y|$, then obviously $|Y| = |X|$). How about transitivity? Well if $|X| = |Y|$ and $|Y| = |Z|$, then it follows that $|X| = |Z|$ by transitivity of usual equality.
The quotient space contains one equivalence class for each whole number, because those are precisely the cardinalities of finite sets. (Note $|\emptyset| = 0$.) So I can identify those and write $\mathcal{F}/$Ⓑ $= \{0, 1, 2, 3, \ldots\}$.

(d) *Suppose an equivalence relation on a sphere identifies all points on the equator with each other, and otherwise each point is alone in its equivalence class. Describe the quotient space, using pictures.*

The equator is one single equivalence class, so it is represented by one point in the quotient space. The upper hemisphere and the lower hemisphere are preserved and separate. This produces a "double sphere," such as in this picture.



(3) [8 mins] *Here is an incorrect proof that relations that are symmetric and transitive must be reflexive.*

Suppose a relation $*$ is symmetric and transitive. By symmetry, $x * y$ and $y * x$.
Since $*$ is also transitive, it follows that $x * x$. So the relation is reflexive.

Find all of the errors in this proof.

The first and gravest error is in the use of symmetry– this argument does not give any quantification for the letters x and y, so it is unclear whether the statement is being asserted for all x, y in the source set for the relation, or just for particular x and y. (Neither is correct: symmetry should be the assertion that $x * y \iff y * x$.)

If it was being asserted for all x and y, then this is the only error, because transitivity would indeed imply that $x * x \quad \forall x$, and this verifies reflexivitiy. However, the original error invalidates the proof.

If it was being asserted for a particular x and y, then the conclusion that $x * x$ would only be valid for a particular x, which is insufficient to certify reflexivity. Either way, this proof fails.

(4) [16 mins]

(a) *How many solutions are there to $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_9 + n_{10} = 100$ with all $n_i \in \mathbb{N}$?*

This is straightforward stars & bars. We can set it up with 100 stars ($\star\star\cdots\star$) and nine dividing bars; this will "chunk" the stars into ten groups. Because no two bars occupy the same space, there is always ≥ 1 star in between successive bars, which correctly corresponds to $n_i \geq 1$. Thus we place 9 bars in the 99 spaces between stars, for a total of $\binom{99}{9}$ solutions, or about 1.7 trillion. (Note: there are currently about 1.9 trillion links on Wikipedia.)

(b) *How many solutions to the same equation where all the n_i are odd natural numbers?*

Well, if each n_i is odd, then I can write each one as $n_i = 2k_i - 1$, which transforms $n = 1, 3, 5, \ldots$ into $k = 1, 2, 3, \ldots$. Then I have

$$\sum_{i=1}^{10} n_i = 100 \implies \sum_{i=1}^{10}(2k_i - 1) = 100 \implies \left(\sum_{i=1}^{10} 2k_i\right) - 10 = 100 \implies 2\sum_{i=1}^{10} k_i = 110 \implies \sum_{i=1}^{10} k_i = 55.$$

Since we are seeking natural-number solutions, this has the same stars & bars form as the previous one, and there are $\binom{54}{9}$ solutions.

(c) *How about if all the n_i are natural numbers and at least two of them are required to be ≥ 10?*

Here is one student's really awesome solution:

Note that it is impossible for NONE of the n_i to be ≥ 10, because ten numbers that are at most 9 can't add up to 100. So the only way for this condition to fail is for exactly one number to be ≥ 10. Let's count how many ways for that to happen. First, there are 10 ways to choose which of the n_i will be the one big number. Then for the remaining nine values, they can be anything from 1 to 9, so each of the nine values has nine choices, independently. There are therefore 9^9 ways to make all of these choices. But once you do that, the tenth (big) number is determined! It's however much you need to add up to 100 exactly. Therefore the total number of solutions to the original question is $\binom{99}{9} - 10 \cdot 9^9$.

Hopefully you are curious how many solutions we had to throw out! The probability a solution having exactly one big number is $\dfrac{10 \cdot 9^9}{\binom{99}{9}} \approx .0022$. That is, if you have ten natural numbers adding up to 100, there's a roughly 99.8% chance that at least two of them are ≥ 10.