Name [1 min]

Math 61-02: First Midterm

Instructions: Every answer requires some justification—that's what the white space is for. On proof problems, if you don't have a full proof, give a proof strategy.

(1) (a) [1 min] What is the contrapositive of *If you cheat on this test on Monday, I will murder you on Wednesday*?

(b) [2 min] How do you prove that one set is a subset of another? Give one brief sentence of explanation.

(c) [3 mins] Prove that $A \subseteq B \implies A \times A \subseteq B \times B$.

(d) [5 mins] Name three distinct elements of $\mathcal{P}\left(\mathcal{P}\Big([3] \setminus \{1\}\Big)\right)$. Watch your symbol hygiene!

(e) [8 mins] Consider the statement $R_n : n! > 2^n$. Prove that this is true for infinitely many values $n \in \mathbb{N}$.

(2) [13 mins] Suppose I have a large supply of sugarcubes and 10 people's coffee mugs, numbered 1–10. A distribution of sugarcubes is just a way of putting some cubes in some mugs. For instance, *put 6 sugarcubes in mug 3, and no cubes in any other mug* is one way of distributing six cubes.

☠ Heads up! If anyone gets four or more sugarcubes, they will DIE of sugar shock. ☠

Let M be the largest number of sugarcubes I can distribute while not killing anyone.

Compute M.

How many ways to....

distribute (any number of) sugarcubes into mugs so that nobody dies?

distribute exactly M sugarcubes into mugs so that nobody dies?

distribute exactly $M - 2$ sugarcubes so that nobody dies?

(3) [21 mins] Let's write $x \equiv y \pmod{n}$ if x and y have the same remainder on division by n. So for instance $13 \equiv 8 \pmod{5}$, $4^2 \equiv 0 \pmod{8}$, and $m + kn \equiv m \pmod{n}$.

(a) Using your knowledge of binomial expansions, show that

$$\forall p \text{ prime}, \quad \forall a, b \in \mathbb{Z}, \qquad (a+b)^p \equiv a^p + b^p \pmod{p},$$

and explain why this is called "the freshman's dream." (If you can't do it for all p, do it for $p = 2$ and 3.)

(b) On the other hand, the corresponding statement for natural numbers,

$$\forall n \in \mathbb{N}, \quad \forall a, b \in \mathbb{Z}, \qquad (a+b)^n \equiv a^n + b^n \pmod{n},$$

is **false**. Therefore its negation is true. (Thank you, Aristotle!)

State the negation and prove it.