MATH 61-02: PRACTICE PROBLEMS FOR FINAL EXAM

(FP1) The *exclusive or* operation, denoted by $\oplus$ and sometimes known as XOR, is defined so that $P \oplus Q$ is true iff P is true or Q is true, but not both. Prove (through a truth table, or otherwise) that for any statements P, Q, R:

(a) $((P \oplus Q) \oplus R) \Leftrightarrow (P \oplus (Q \oplus R))$

(b) $(P \wedge (Q \oplus R)) \Leftrightarrow ((P \wedge Q) \oplus (P \wedge R))$

(FP2) Prove the following:

(a) $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{N}$

(b) $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for $n \geq 2$

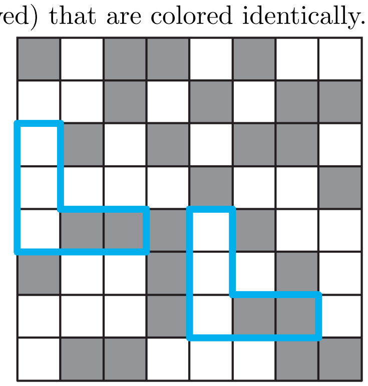
(c) If $A_1, A_2, \ldots, A_n, B$ are sets ($n \geq 1$), then $\bigcup_{i \in [n]} (A_i \setminus B) = (\bigcup_{i \in [n]} A_i) \setminus B$.

(d) Let F_n denote the nth term of the Fibonacci sequence (where $F_1 = 1, F_2 = 1$, and $F_k = F_{k-2} + F_{k-1}$ $\forall k \geq 3$). Show that $\forall n \in \mathbb{N}$, $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.

(FP3) (a) For a fixed natural number $m \geq 2$, let's write $\mathbb{Z}_m$ for the quotient space $\mathbb{Z}/\equiv_m$ of equivalence classes mod m. Consider the map $f : \mathbb{Z}^n \to (\mathbb{Z}_m)^n$ given by taking the remainder of each coordinate mod m, so for instance if $m = 4$ and $n = 3$, we have $f((4, 10, 2)) = (0, 2, 2)$. If A is a subset of $\mathbb{Z}^n$, how big must its cardinality be (in terms of m and n) in order to ensure that $|f(A)| < |A|$?

(b) Everyone in a group of people is asked to name their favorite letter of the English alphabet, their favorite integer between 1 and 10 inclusive, and their birthday. How large does the group need to be to ensure that there are at least five people who give identical responses?

(c) The squares of an 8×8 grid are colored black or white. Let's use the term *L-region* for 5 squares arranged in an L, as shown in the picture (note orientation matters: the corner of the L must be in its lower left). Prove that no matter how we color the grid, there must be two distinct L-regions (partial overlap allowed) that are colored identically. See example below.



(FP4) Fix n, and for any function $f : [n] \to [n]$, define $N(f) := \prod_{i=1}^{n} (i - f(i))$.

(a) If $n = 5$ and f is the constant map $f(x) = 1$, compute $N(f)$.

(b) Give necessary and sufficient conditions on f for $N(f) \neq 0$.

(c) For $n = 4$, give an example of a bijection f with $N(f) > 0$.

(d) (harder) If n is odd, prove that f is a bijection $\implies N(f)$ is even.

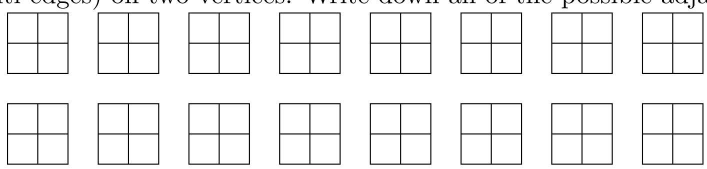
(FP5) (a) I have 30 sugarcubes, and there are 10 coffee mugs lined up in a row. How many ways are there to distribute all of the sugarcubes into mugs? Remember that four sugarcubes causes death. If

I distribute the cubes randomly (making all distributions equally likely), what is the probability of murder?

(b) 42 chairs are set up in a row for the Discrete Math garlic-eating contest. Only six people show up, but none of them will sit next to each other. In how may ways can the eaters be seated if
  (i) the eaters aren't allowed to sit in the chairs on the ends of the row?
  (ii) the eaters are allowed to sit anywhere they want (but still refuse to sit next to each other)?

(c) I shuffle a standard pack of 52 cards and cut them into two equal stacks of 26 cards. I'm astounded to find that one stack is all red cards and the other stack is all black! What're the odds?

(FP6) Let A be a finite set with cardinality n.
(a) Explain why the number of relations on A is 2^{n^2}.
(b) For $A = \{1, 2\}$, there are sixteen relations on A. You can record them as directed graphs (with loops but no multi-edges) on two vertices. Write down all of the possible adjacency matrices.



(c) Which matrix corresponds to the relation $R = \{(1, 1), (2, 1)\}$? Draw the corresponding digraph.
(d) Suppose a relation on $A = \{1, 2\}$ is selected uniformly at random. What is the probability that the relation is symmetric?
(e) Now consider the general case, $|A| = n$. What is the probability that a random relation is symmetric? Anti-symmetric?

(FP7) (a) Suppose that $(X, \leq_1)$ and $(Y, \leq_2)$ are posets. Show that $(X \times Y, \leq)$ is a poset where $(a, b) \leq (c, d)$ iff $a \leq_1 c$ and $b \leq_2 d$. We can call that the *product poset*.
(b) Give an example of a pair of comparable elements and a pair of non-comparable elements in the product poset if $X = \{1, 2, 5\}$, $Y = \{3, 6\}$ and both $\leq_1$ and $\leq_2$ are the standard less-than-or-equal relation on integers.

(FP8) Let $f : X \to Y$ be a function, and let $S, T \subseteq X$ and $A, B \subseteq Y$. Furthermore, for $C \subseteq Y$ recall that $f^{-1}(C) = \{x \in X \mid f(x) \in C\}$. Prove that:
(a) $f(S \cup T) = f(S) \cup f(T)$.
(b) $f(S \cap T) \subseteq f(S) \cap f(T)$.
(c) If f is injective, then $f(S \cap T) = f(S) \cap f(T)$.
(d) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
(e) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(FP9) (a) Write out the quantified definitions. (For example, $A \subseteq B$ iff $\forall x \in A,\ x \in B$.)
  (i) A relation R on X is reflexive/symmetric/transitive/antisymmetric iff...
  (ii) A relation f from X to Y is a function iff... (Note that the notation for "*there exists a unique*" is "$\exists!$".)
  (iii) A function $f : X \to Y$ is injective/surjective iff...
(b) How would you prove a function is injective? How would you prove a function is surjective?
(c) If $f : X \to Y$ and $g : Y \to X$ satisfy $g \circ f = Id_X$, show that f is injective and g is surjective.
(d) Suppose you are given a bijection $f : X \to Y$. Give an explicit bijection $g : \mathcal{P}(X) \to \mathcal{P}(Y)$. (In other words, for a subset $A \subseteq X$, you should be able to write down $g(A)$ in set-builder notation, using f.)

(FP10) (a) Show that the relation $\sim$ on $\mathbb{N}$ by $a \sim b$ iff $\exists n \in \mathbb{N}$ such that $ab = n^2$ is an equivalence relation.
(b) Prove that for $m \in \mathbb{N}$, we have $m \in [6]$ iff, when we break down m into its prime decomposition, the exponent of 2 and the exponent of 3 are odd, while all the other exponents are even.

(c) Describe [16]. Can you describe in general what the equivalence class $[N]$ looks like for an arbitrary $N \in \mathbb{N}$?
(d) Let $A = \mathbb{N}/\sim$ be the set of equivalence classes. Show that A is countably infinite.

(FP11) Prove that $A = \mathbb{Q} \cap (0, 1)$ and $B = \mathbb{Q} \cap [6, 60]$ have the same cardinality.

(FP12) Let Q be the set of real numbers which are solutions to quadratic equations $ax^2 + bx + c = 0$ with integer coefficients (so $a, b, c \in \mathbb{Z}$).
(a) Show that $\mathbb{Q} \subseteq Q$, but $Q \not\subseteq \mathbb{Q}$.
(b) Recall that the quadratic formula ensures that a quadratic equation has at most two distinct integer roots. Show that Q is countable by describing a plan of how to enumerate it, and give the first few terms in your list.

(FP13) A graph G is called *bipartite* if $V(G)$ can be partitioned into two sets S and T such that every edge of G is incident to one vertex in S and one vertex in T. A *complete bipartite graph* is a bipartite graph where every possible edge is present.
(a) For what values of n are K_n, C_n, W_n bipartite? (Recall that K_n is the complete graph on n vertices, C_n is the cycle on n vertices, and we'll write W_n for the "wheel" graph with $n+1$ vertices, formed by connecting a central vertex to every vertex of a C_n.)
(b) Must bipartite graphs be simple?
(c) A complete bipartite graph where $|S| = m, |T| = n$ is denoted $K_{m,n}$. Draw a $K_{3,3}$ and a $K_{4,2}$. What are $|V(K_{m,n})|$ and $|E(K_{m,n})|$ for general m, n?
(d) A *tree* is a connected graph with no cycles. Prove that trees are bipartite. (Begin with an example of a tree and figure out what S and T should be in your example.
(e) (harder) Prove that a graph is bipartite if and only if it does not contain any odd cycles.

(FP14) (a) Prove by induction that a connected graph with n vertices must have at least $n - 1$ edges.
(b) Prove that a tree with at least one edge must contain at least 2 vertices of degree 1. (Such a vertex is called a *leaf*.)

(FP15) (harder—but fun) Here's a famous problem: Suppose you place n points on a circle, then join every pair of points with a chord. Let $f(n)$ be the number of regions that the disk is cut into, supposing no more than two edges cross at a single point in the diagram. For example, when you place two points on a circle boundary and draw the chord between them, that cuts the disk into two parts, so $f(2) = 2$; we can also check that $f(3) = 4$ because the chords form a triangle inscribed in a circle.
(a) Compute the values of $f(4)$ and $f(5)$, with pictures. Conjecture what $f(n)$ is in general.
(b) Compute $f(6)$. Does this fit with your conjecture?
(c) Alright, so your conjecture didn't work. We'll find an explicit formula for $f(n)$ by using several steps. First, a *planar graph* is a graph that can be drawn in the plane in such a way that its edges intersect only at their endpoints. It is a general fact about planar graphs that $V - E + R = 1$, where R is the number of regions that the graph cuts the plane into. (Try this on a few examples to convince yourself.) That means you can count the number of regions by $R = E - V + 1$. Let's interpret our disk-and-chords picture as a planar graph by placing extra vertices at all edge crossings!
  (i) How many vertices does this graph have? Include all of the points that you placed on the boundary plus the extra vertices at the edge crossings. Along the way, explain why there are $\binom{n}{4}$ edge crossings.
  (ii) Show that the number of edges in the planar graph is $n + \binom{n}{2} + 2\binom{n}{4}$.
  (iii) Put this together to get a general formula for $f(n)$. Does this now fit with what you computed for $f(6)$?
(d) Considering your formula, use Pascal's Triangle to explain: Why did $f(n) = 2^{n-1}$ for the first few terms and then stop? Will they ever be equal again?