MATH 61-02: WORKSHEET 9 (§6.1-6.4)

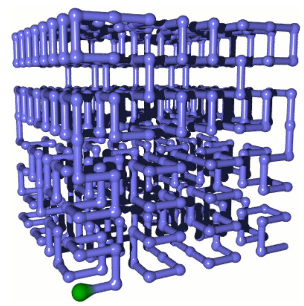
(W1) Show that $\mathbb{Z}^3$ is countable.

Answer. One solution is as follows: We know that $\mathbb{Z}$ is countable since we can enumerate it in a list that alternates between positive and negative integers: $0, 1, -1, 2, -2, 3, -3, \ldots$.

If we insist (and we should not), we can write this as a function $f : \mathbb{N} \to \mathbb{Z}$,

$$f(x) = \begin{cases} \frac{x-1}{2}, & x \equiv 1 \pmod 2 \\ -\frac{x}{2}, & x \equiv 0 \pmod 2 \end{cases}$$

Now we know that the Cartesian product of two countable sets is countable, and since $\mathbb{Z}$ is countable, $\mathbb{Z}\times\mathbb{Z} = \mathbb{Z}^2$ is countable. Finally, there is clearly a bijection $\mathbb{Z}^2\times\mathbb{Z} \to \mathbb{Z}^3$ given by $((a, b), c) \mapsto (a, b, c)$, so since $\mathbb{Z}^2 \times \mathbb{Z}$ is countable, $\mathbb{Z}^3$ is too.

Alternately, we can snake-enumerate the lattice points: 

(W2) Suppose that you're walking on a road and every mile you go you come to another fork in the road. Suppose the roads go on forever. Consider the set P of all paths (that is, all choices of L or R at every fork; so one path is LLLLLLL... and another path is LRLRRLRRRLRRRR....). Prove that $\aleph_0 < |P| \leq c$, where $\aleph_0 = |\mathbb{Z}|$ and $c = |\mathbb{R}|$. (Bonus: prove $|P| = c$.)

Answer. It is clear that $|P| \geq \aleph_0$, simply because it is infinite. Now let's show it's strictly bigger by showing that it is not countable. Suppose to the contrary that we COULD enumerate all the paths in P in a list $p_1, p_2, p_3, \ldots$ If we take the first letter of p_1, the second letter of p_2, the third letter of p_3, and so on, we can form a new string of L/R characters by changing every L to an R and vice versa. The new path differs from each p_i in the ith position, so it not on the list, which means that the list was incomplete. So we've shown $|P| > \aleph_0$.

Now let's show that $|P| \leq c$, by showing that there's an injection from $P \to \mathbb{R}$. I can do this by putting a decimal point first, and then converting L/R characters to $0/1$ digits. That turns every string of L and R into a real number (for instance, it converts the sequence of all L into $.111\bar{1}$, which equals $1/9$ in decimal).

Bonus: To show that it has the *same* cardinality as the reals, I can just interpret the $0/1$ digits in binary, and I get all of $[0, 1]$, which has cardinality c. (Note that there are some details to fuss with here, because $.10000\bar{0} = .01111\bar{1}$, but the redundancy is countable and so not that serious.)

(W3) Prove that $|(0, 1)| = |[0, 1]|$, or in other words, there is a bijection between the open interval and the closed interval. (Hint: by Schroeder-Bernstein, it suffices to find injections both ways.)

Answer. First direction: Since $(0, 1)$ is a subset of $[0, 1]$, the inclusion $f(x) = x$ is an injection $(0, 1) \to [0, 1]$.

Second direction: The function $g(x) = \frac{1}{4} + \frac{1}{2}x$ is an injection $[0, 1] \to (0, 1)$ with image $[\frac{1}{4}, \frac{3}{4}]$.

Since there are injections both ways, Schroeder-Bernstein guarantees the existence of a bijection.

(W4) **Bonus question (extra credit):** A monster is moving in a straight line in $\mathbb{R}^2$ at a constant speed so that every minute, they arrive at a lattice point. Suppose every minute you can place one bomb at a lattice point that will kill the monster if it is there. You don't know where the monster started or where it is. Can you strategically place bombs in such a way that is guaranteed to eventually kill the monster?

Answer. We can parametrize the movement of the monster: If the monster starts at the lattice point (a, b) and translates by $(c, d) \in \mathbb{Z}^2$ every minute, then its position at time t will be $(a + ct, b + dt)$. But $|\mathbb{N}| = |\mathbb{Z}^4|$ by an argument analogous to that of (W1), so there exists a bijection $f : \mathbb{N} \to \mathbb{Z}^4$, and at a given time t, if $f(t) = (w, x, y, z)$, we can bomb the lattice point $(w + yt, x + zt)$. Since a, b, c, d are fixed integers and f is surjective, there must exist a time t for which $f(t) = (a, b, c, d)$, at which point we will kill the monster!