MATH 61-02: WORKSHEET 7 (§5.1-5.2)

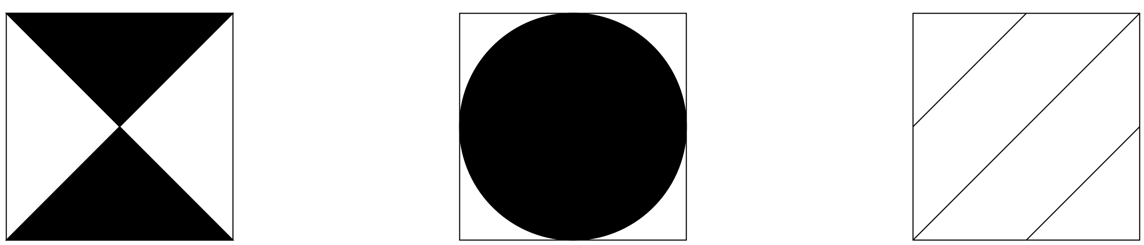
(W0) The extra credit problem from WS5: How many tickets do you need to buy to ensure that one of your tickets will have at least five correct predictions? Prove that the answer you give is the minimum number of tickets for which this is true.

Answer. Three tickets suffice: Suppose we buy one ticket betting all H, one ticket betting all D, and one ticket betting all A. Since there are thirteen matches with three possible outcomes each, by the Pigeonhole Principle one of the outcomes will be the result of at least five matches, and therefore one of our tickets will have at least five correct predictions. If we buy only one or two tickets, then for each match there must be an outcome which we did not bet for on any of our tickets, and so it is possible that none of our tickets will have any correct predictions, so three tickets is the minimum number of tickets for which this is true.

(W1) Let R be the relation on the interval $[0,1]$ defined by xRy if and only if $x^2 \leq y$. Sketch the graph of R. Decide if it is reflexive, symmetric, transitive, and/or anti-symmetric.

Answer. A sketch for the graph of R can be obtained easily on a graphing calculator. R is reflexive, since $x^2 \leq x$ for all $0 \leq x \leq 1$. R is not symmetric, since $0^2 \leq 1$, but $1^2 = 1$ is clearly greater than 0. R is not antisymmetric, since $0.4^2 \leq 0.6, 0.6^2 \leq 0.4$ but $0.4 \neq 0.6$. R is not transitive, since $0.6^2 \leq 0.4, 0.4^2 \leq 0.2$, but $0.6^2 = 0.36 > 0.2$.

(W2) For each of the following relations on $[0,1]$, decide whether it is reflexive, symmetric, and/or anti-symmetric.
(Bonus: also decide whether each is transitive, and use that to identify which are equivalence relations and which are partial orders.)



Answer. Call the relations R, S, T corresponding to the pictures from left to right. For R, we will assume that the boundaries of the triangles given by $y = x$ and $y = 1 - x$ are entirely contained in R. Then, R is clearly reflexive. R is not symmetric, since $(0.5, 1) \in R$ but $(1, 0.5) \notin R$. R is not antisymmetric, since $(0,1), (1,0) \in R$ but $0 \neq 1$.

For S, we will assume that the boundary of the circle $(x - 0.5)^2 + (y - 0.5)^2 = (0.5)^2$ is entirely contained in S. S is not reflexive, since $(0,0) \notin S$. S is symmetric, since the circle is symmetric across the line $y = x$, so if a point $(a,b) \in S$, then the corresponding point $(b,a) \in S$. S is not antisymmetric, however, since the points $(0.4, 0.6), (0.6, 0.4) \in S$, but $0.4 \neq 0.6$.

T is reflexive, since all points (a,a) for $0 \leq a \leq 1$ are clearly contained in T. T is symmetric, since the graph of T is symmetric across the line $y = x$, so if a point $(a,b) \in T$, then the corresponding point $(b,a) \in T$. T is not antisymmetric, however, since the points $(0.25, 0.75), (0.75, 0.25) \in T$, but $0.25 \neq 0.75$.

(W3) Consider the relation on A=$\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ given by $(a,b)F(c,d) \iff ad = bc$. Check that it is an equivalence relation. Give several elements of the equivalence classes $[(0,1)]$ and $[(2,3)]$ in the associated partition. Sketch A and show a few blocks of the partition in your picture.

Answer. Firstly, we can realise that this equivalence relation is really just that for determining when two fractions are equivalent, since $ad = bc$ iff $\frac{a}{b} = \frac{c}{d}$, so $(a,b)F(c,d) \iff \frac{a}{b} = \frac{c}{d}$. Now, this inherits all the relation properties of equality in the real numbers, so this is clearly an equivalence relation as equality is an equivalence relation.

Phrased in this light, the equivalence classes $[(0,1)]$ and $[(2,3)]$ consist of all pairs (a,b) such that $\frac{a}{b} = 0$ and $\frac{a}{b} = \frac{2}{3}$ respectively. In general, an equivalence class $[(c,d)]$ consists of all lattice points (a,b) lying on the line $y = \frac{d}{c}x$, and a sketch of the blocks of the partition should reflect this.

(W4) Give a relation R on $A = \{blue, red, yellow, green\}$ that is reflexive but not transitive. What is the smallest number of elements $|R|$ required to achieve this?

Answer. Such a relation is $R = \{(blue, blue), (red, red), (yellow, yellow), (green, green), (blue, red), (red, yellow)\}$. The minimum $|R|$ required to achieve this is 6: for R to be reflexive, it must contain the four elements $(blue, blue), (red, red), (yellow, yellow), (green, green)$. If R consisted of only these four elements, it is clearly transitive: The only way that two elements (a,b) and (b,c) can be in R is if $a = b = c$, in which case (a,c) is also in R. If R had five elements consisting of the previously mentioned four elements plus a fifth element (x,y), where $x, y \in A$ and $x \neq y$, then R is still transitive: The only additional ways (aside from the ones covered above) that two elements (a,b) and (b,c) can be in R is if $a = b$ and (b,c) is the fifth element we added, in which case (a,c) is the fifth element we added and so must be in R. Since we have given an R satisfying the conditions above such that $|R| = 6$, it is now clear that this is the smallest number of elements $|R|$ required to achieve this.