

MATH 61-02: WORKSHEET 5 (§4.1-§4.3)

- (W1) (a) Everyone can choose which primary they go to, so each person has a total of $4+3 = 7$ candidates to choose from. Each person makes their choice independently, so the total number of ways the class can vote is 7^{42} .
- (b) The number of ways to rank the candidates is $7!$, so the number of ways the class can rank them is $(7!)^{42}$. Kris is grievously wrong, probably because he added $7!$ to itself 42 times instead of multiplying it with itself. This is an AND situation rather than an OR situation, because everyone is making a choice—that's why you multiply.
- (c) Anyone going to the Democratic primary has $3!$ possible preference rankings, and anyone going to the Republican primary has $\binom{4}{2}$ possible ways to choose their two favourite candidates unranked. Since you only get to go to one primary, it's an OR situation, and the number of ways a single person can vote is $3! + \binom{4}{2}$. Since each person makes their choice independently, the total number of ways the class can vote is $(3! + \binom{4}{2})^{42} = 12^{42}$.
- (d) Anyone going to the Democratic primary has $3!$ possible Democratic rankings and 4 possible least favourite Republican candidates. Anyone going to the Republican primary has $4!$ possible rankings and 3 possible least favourite candidates. Since each person makes their choice independently, the total number of ways the class can vote is $(3! \times 4 + 4! \times 3)^{42} = (24 + 72)^{42} = 96^{42}$.
- (W2) (a) For each match, we can choose to bet on Home, Draw, or Away. There are 3 choices for 13 independent matches, so there are 3^{13} different tickets we can buy.
- (b) If no two consecutive matches will have the same outcome, we have 3 choices to bet on for match 1, and then 2 choices to bet on for matches 2-13, since we can't bet on whatever we bet on in the previous numbered match. Hence there are $3 \cdot 2^{12}$ different tickets we can buy.
- (c) We can pick 3 matches out of 13 to mark H in $\binom{13}{3}$ ways. For the remaining matches, we can freely choose D or A, so there are 2 choices to bet on for each of these 10 matches. Hence there are $\binom{13}{3} \cdot 2^{10}$ different tickets we can buy.
- (d) We can pick which matches to mark H in exactly $\binom{13}{3}$ ways. For the remaining 10 matches, we can pick which matches to bet A in exactly $\binom{10}{6}$ ways. This will uniquely determine the ones to mark D. Hence there are $\binom{13}{3} \binom{10}{6}$ different tickets we can buy. (Note that there are several possible ways to answer this that look different but come out the same.)
- (e) First, we can pick 7 H's in $\binom{13}{7}$ ways, leaving a 2-way choice in the remaining 6 matches. Similarly, we can bet H eight times in $\binom{13}{8}$ ways, leaving a 2-way choice in the remaining 5 matches. This gives us $\binom{13}{7} \cdot 2^6 + \binom{13}{8} \cdot 2^5$ different tickets we can buy.
- (f) We can bet whatever we want for matches 1-7; being a palindrome means that this determines the bets in matches 8-13. Hence there are 3^7 different palindromic bet strings.
- (g) Since this bet string needs to be a palindrome, it is determined by the bets in matches 1-7, just like in the last part. First consider how to get 7 H's. Since this is an odd number, the middle bet must be an H, and three of the first six must be H as well—then we can pick A or D freely in the remaining three. That makes $\binom{6}{3} \cdot 2^3$ possibilities. Similarly, if we bet that the home team will win eight matches, we need to bet that the home team will win four of the first six matches, and for the other two of the first six, plus match 7, we can choose freely. Hence there are $\binom{6}{3} \cdot 2^3 + \binom{6}{4} \cdot 2^3$ different tickets we can buy.
- (h) For matches 3, 6, 9, and 12, we know the away team will win. For matches 2, 5, 7, 11, and 13, we have two choices: the away team will win or the teams will draw. (Notice that 3 is prime, but we already know the outcome for that match.) For the remaining matches - 1, 4, 8, and 10 - we do not know anything and still have three outcomes to choose from. Hence we have $2^5 \cdot 3^4$ different tickets we can buy.
- (i) I'll give you guys the solution in a few weeks :)

- (W3) (a) (i) Label the points $1-n$. Then vertex 1 can form a diagonal with vertices $3, 4, \dots, n-1$, which means there are $n-3$ diagonals coming out of it, and therefore out of each of the n vertices. But this will count each diagonal twice, so we must divide by 2 to correct the overcounting. Hence there are $\frac{n(n-3)}{2}$ diagonals in a convex n -gon.
- (ii) Let D_n be “there are $\frac{n(n-3)}{2}$ diagonals in a convex n -gon”; we want to prove $D_n \quad \forall n \geq 3$. We already discussed $n=3$ and $n=4$, which handles the base case. Now suppose we knew that for a convex k -gon, where $k \geq 3$, there are $\frac{k(k-3)}{2}$ diagonals. Take a regular $(k+1)$ -gon, and number its vertices 1 through $k+1$. Note that the vertices 1 through k form a k -gon; it’s not regular, but that does not affect its number of diagonals. Color those diagonals red just to keep track of them. How many additional diagonals does the $(k+1)$ -gon have? It’s got the one between vertex 1 and vertex k , plus the ones that connect vertex $k+1$ to each of the vertices numbered 2 through $k-1$; that is $k-1$ new diagonals in all. But now, our $(k+1)$ -gon has $\frac{k(k-3)}{2} + (k-1) = \frac{k^2-3k+2k-4+2}{2} = \frac{k^2-k-2}{2} = \frac{(k+1)(k-2)}{2}$ diagonals, so induction is complete and we have the desired result.
- (b) This problem just boils down to picking two rows and picking two columns! Then you select the four vertices at the intersections of the rows and the columns. There are clearly $\binom{m}{2} \cdot \binom{n}{2}$ ways to do this.