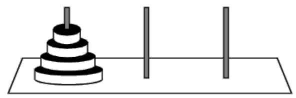
MATH 61-02: WORKSHEET 4 (§2.2-2.3)

(W1) The *Towers of Hanoi* is a game consisting of 3 rods and n disks of different sizes, looking like this:



The disks start out in ascending size order on the left rod (as in the figure). The aim is to move all disks to the right rod while obeying the following rule: each move consists of taking the one top disk from one rod and moving it to another rod, while never placing a disk on top of a smaller disk. Practice with 2, 3, and 4 disks online (http://vornlocher.de/tower.html).

Prove by induction: If there are n disks, the game can be won in $2^n - 1$ moves.

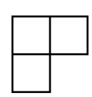
(Bonus: convince yourself this is optimal.)

Answer. We prove that this is possible by induction on the number of disks. For $n = 1$, it is clear that we can win the game in one move (by moving the one disk to the right rod on the first move). Now, suppose that for $n = k$ disks, we knew that the game could be won in $2^k - 1$ moves, and suppose that we played with $k + 1$ disks. We can win the game through moving the top k disks to the centre rod, moving the bottom disk from the left rod to the right rod, and then moving the k disks from the second rod to the right rod. By the inductive hypothesis, we can move the k disks to the second rod in $2^k - 1$ moves (through changing any one of our moves involving the right rod to instead involve the centre rod, and vice versa), and it takes one move to move the bottom disk from the left rod to the right rod, so we can win this game in $2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$ moves. Hence, induction is complete and so if there are n disks, the game can be won in $2^n - 1$ moves.

(W2) Suppose we place five points in a square of area 16. Prove that there exist two points which are a distance of at most $2\sqrt{2}$ apart. Can you come up with a similar statement for some number of points in a pentagon?

Answer. For the square, if we divide the square of area 16 - hence of side length 4 - into four smaller squares of side length 2, we have four regions and hence by the Pigeonhole Principle, there must be two points inside the same region. The maximum distance between any two points in a square is achieved when the two points are at opposite corners of the square, so if there are two points inside the same square of side length 2 it is clear that these two points must be a distance of at most $2\sqrt{2}$ apart.

It is possible to come up with a variety of statements for a pentagon. One such statement is as follows: If we place six points in a regular pentagon of side length 1, there are two points which are a distance of at most 1 apart from each other.

(W3) An *L-triomino* is an L-shaped cluster of three squares, like this: 

Prove that for all $n \in \mathbb{N}$, a $2^n \times 2^n$ checkerboard with any one square removed can be tiled with L-triominoes. (You can email kristofer.siy@tufts.edu for a hint on this one if you're stuck.)

Answer. We prove that this is possible by induction on n. For $n = 1$, it is clear that no matter which square we remove from our 2×2 checkerboard, the remaining three squares can be covered with a single L-triomino. Now, suppose that we knew we could cover a $2^k \times 2^k$ checkerboard with any one square removed by L-triominoes, and suppose we had a $2^{k+1} \times 2^{k+1}$ checkerboard with any one square removed. If we divide our checkerboard into four $2^k \times 2^k$ checkerboards, this one square must lie in precisely one quadrant (i.e., one of these $2^k \times 2^k$ checkerboards). Consider the centre 2×2 square of our $2^{k+1} \times 2^{k+1}$ checkerboard containing one square from each of the four quadrants. We can remove the three squares in this 2×2 square from the three quadrants that our removed square did *not* come from. But now by the inductive hypothesis, each of our four $2^k \times 2^k$ checkerboards has one square removed and so can be tiled by L-triominoes, and we can add one more L-triomino in the place of the three squares we removed from the centre 2×2 square so that every square of our checkerboard except the single square we removed at the beginning is tiled by an L-triomino. Hence induction is complete, so we must have that for all $n \in \mathbb{N}$, a $2^n \times 2^n$ checkerboard with any one square removed can be tiled with L-triominoes.

(W4) I went to this crazy party last weekend where they managed to cram all 5200 undergraduates at Tufts into a single house. Of course, since we didn't all know each other, some people shook hands to introduce themselves. I want to prove that there were two people at the party who shook hands with the same number of people. One possible solution starts like this: *Suppose not. Let's number the people 1 through 5200, and let $h(n)$ denote the number of people that person n shook hands with. Then the numbers $h(1), h(2), \ldots, h(5200)$ are 5200 different integers between 0 and 5199.* Finish the proof. (There are a few more steps to go.)

Answer. Suppose not. Let's number the people 1 through 5200, and let $h(n)$ denote the number of people that person n shook hands with. Then the numbers $h(1), h(2), \ldots, h(5200)$ are 5200 different integers between 0 and 5199. But there are only 5200 different integers between 0 and 5199 inclusive, so the numbers $h(1), h(2), \ldots, h(5200)$ are all different. In particular, there exist some people numbered p and q such that $h(p) = 0$ and $h(q) = 5199$. But by how we defined $h(n)$, this implies that the person numbered p shook hands with nobody, and the person numbered q shook hands with everybody except themselves, so p and q both shook hands and didn't shake hands with each other, which is a contradiction. So it must be true that there are two people at the party who shook hands with the same number of people.