MATH 61-02: WORKSHEET 3 (§2.1)

(W1) Let S be a subset of $[9]$ such that $|S| = 6$. Show that $\exists a, b \in S$ s.t. $a + b = 10$.

Answer. Case 1: ($5 \in S$) In this case we are done, because $5 + 5 = 10$. (The setup does not require $a \neq b$.)
Case 2: ($5 \notin S$) The possible numbers are now partitioned as $\{\{4,6\},\{3,7\},\{2,8\},\{1,9\}\}$. As soon as I have both numbers from one of these pairs, I have numbers that sum to 10. Suppose I can choose numbers from $[9]$ without getting a pair that sum to ten. Then there is at most one from each of the four pairs, so at most four numbers in the set. Since S has 6 elements, this is impossible. (In the language of §2.3, this is an application of the pigeonhole principle.)

(W2) Consider the sequence 7, 87, 887, 8887, ... Show that no number in this sequence is the sum of three squares.

Answer. Observe that the square of any integer will have a remainder of 0, 1, or 4 when divided by 8. (If you need to convince yourself of this, expand the expressions $(8k)^2$, $(8k+1)^2$, ...) It is then easy to see by exhausting all possible combinations that no matter what three integers we take, when we take the sum of their squares and divide it by 8 it is impossible to get a remainder of 7, but all numbers in the given sequence have a remainder of 7 when divided by 8, so no number in this sequence is the sum of three squares.

(W3) We are going to play a game where I think of a number between 1 and N; each time you guess a number, I will tell you if it is high or if it is low. Suppose N is at least 1000. Come up with a strategy that will guarantee you find the number in fewer than $N/10$ steps.

Answer. We present one optimal strategy. After each new round of information, the possible mystery numbers range from some a to some b; on each turn we guess a number roughly in the middle, namely $\lfloor \frac{a+b}{2} \rfloor$, which leaves us with a new and smaller range of integers where our number must be. This process repeats until we get it right or have only one number in our possible range, at which point we know it must be our number. Let's call this "the halfsies strategy." Let P_n be the statement: *if you must guess from a range of at most 2^n consecutive numbers, then the halfsies strategy always succeeds in at most n steps.*
Base case (P_1): Suppose you consider a list of at most two consecutive numbers, so the list is either $\{a\}$ or $\{a, a+1\}$. Then the halfsies strategy tells you to guess a, and that's either right or too low, and either way you know the mystery number. ✓
Domino step ($P_k \implies P_{k+1}$): Assume that for up to 2^k consecutive numbers, you can be sure to win the game with k guesses. Now consider a list $\{a, a+1, a+2, \ldots, b\}$ of at most 2^{k+1} consecutive numbers, and let's find a sure winning strategy. Let your first guess be in the middle; if it's wrong, it narrows down the choices to at most half as many, which means the number of remaining choices is from a set of no more than 2^k consecutive numbers. Now apply the hypothesis P_k to conclude that we can finish with at most k more guesses. ✓

Thus we have shown P_n for all $n \in \mathbb{N}$. How many steps did it take? It took n steps, so we need to solve $N \leq 2^n$ for n. Taking log base two of both sides shows that $n = \lceil \log_2 N \rceil$ suffices. And certainly $\log_2 N$ is much smaller than $N/10$ for $N \geq 1000$.

(W4) (a) Find all solutions to the equation $x^2 = 3y + 2$ for $x, y \in \mathbb{N}$. (With proof, of course.)
(b) (Bonus—Extra Credit) Prove that $x^2 + y^2 = 3z^2$ has no solutions $x, y, z \in \mathbb{N}$.

Answer. (a) We claim there are no solutions to the given equation. First, notice that when a natural number has a remainder of 0 when divided by 3, its square must also have a remainder of zero when divided by 3, and when a natural number has a remainder of 1 or 2 when divided by 3, its square must have a remainder of 1 when divided by 3. (To see this, expand $(3k)^2, (3k+1)^2, (3k+2)^2$ and look at the terms not divisible by 3.) But now, observe that the equation asks us to find a natural number such that its square is two more than an integer multiple of 3 - that is, it must leave a remainder of 2 when divided by 3. Our casework above proves this is impossible.

(b) Suppose that there did exist $x, y, z \in \mathbb{N}$ so that $x^2 + y^2 = 3z^2$. In particular, take a solution $x_0^2 + y_0^2 = 3z_0^2$ with z_0 as small as possible (that is, there do not exist any solutions with $z < z_0$). By our work above, the square of any integer must leave a remainder of 0 or 1 when divided by 3. Furthermore, the right hand side of the equation is divisible by 3, so the left hand side of the equation is also divisible by 3. Observe that the only way this is possible is if both x_0 and y_0 are divisible by 3. Then, $x_0 = 3m$ and $y_0 = 3n$ for some $m, n \in N$. It then follows that

$$\begin{aligned} x_0^2 + y_0^2 &= 3z_0^2 \\ \implies (3m)^2 + (3n)^2 &= 3z_0^2 \\ \implies 9m^2 + 9n^2 &= 3z_0^2 \\ \implies 3(m^2 + n^2) &= z_0^2, \end{aligned}$$

and clearly, it must be the case that z_0 is divisible by 3, so $z_0 = 3q$ for some $q \in \mathbb{N}$. But then we have that

$$\begin{aligned} 3(m^2 + n^2) &= z_0^2 \\ \implies 3(m^2 + n^2) &= (3q)^2 \\ \implies 3(m^2 + n^2) &= 9q^2 \\ \implies m^2 + n^2 &= 3q^2, \end{aligned}$$

which gives us a solution to the equation where $q < z_0$, which contradicts the minimality of z_0. Hence there are no solutions to the equation.