MATH 61-02: WORKSHEET 2 (§1.3-1.7)

(W1) Let's say that if $a < b$ are the endpoints of interval I, then the length of the interval is denoted $|I|$ and equals $b - a$. For this problem you may use the fact that $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

(a) Give an partition $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}^+$ (the positive real numbers) in which each $B_n \in \mathcal{B}$ is an interval, and if $i < j$, then $|B_i| > |B_j|$. Practice using summation notation to explicitly write down the endpoints of your intervals B_n.

(b) Can you give a partition $\mathcal{C}$ of $\mathbb{R}$ with the same properties?

Answer. (a) A partition could be given by $B_1 = (0, 1]$ and $B_n = (\sum_{k=1}^{n-1} \frac{1}{k}, \sum_{k=1}^{n} \frac{1}{k}]$ for $n > 1$.

(b) A partition could be given by $B_1 = [0, 1]$, $B_2 = [-\frac{1}{2}, 0)$, and then $B_{2k-1} = (\sum_{i=1}^{k-1} \frac{1}{2i-1}, \sum_{i=1}^{k} \frac{1}{2i-1}]$, $B_{2k} = [-\sum_{i=1}^{k} \frac{1}{2k}, -\sum_{i=1}^{k-1} \frac{1}{2k})$ for $k > 1$.

It is left as an exercise in both cases to show that the given sets indeed define a desired partition.

(W2) In class, you have been learning the building blocks of logic. In this problem, we explore some of the *replacement rules* of propositional logic. These are steps that allow us to replace a logical sentence with an equivalent one, knowing that the truth value remains intact in the course of the replacement.[1]

(a) One replacement rule says that we can replace $P \wedge (Q \vee R)$ with $(P \wedge Q) \vee (P \wedge R)$. Verify that these two statements are equivalent using a truth table.

(b) Another replacement rule says that $P \Leftrightarrow Q$ can be replaced by $(P \wedge Q) \vee (\sim P \wedge \sim Q)$. Verify that this is true through a truth table, and explain in words why this would be true.

(c) Whoops, my paper fell in the rain and I can't read it anymore. To the best of my knowledge, one rule reads that

$$P \Rightarrow Q \quad \rightsquigarrow \quad \sim P \vee Q$$

and this other one says

$$(P \wedge Q) \Rightarrow R \quad \rightsquigarrow \quad P \Rightarrow (Q \vee R).$$

My friend told me that between the two rules written on my page, exactly one of the logical symbols got wet. Which of the two rules is right?

(Bonus challenge problem: try to fix the other one by changing one symbol!)

Answer. (a) The truth table should read:

P	**Q**	**R**	$P \wedge (Q \vee R) \quad \rightsquigarrow \quad (P \wedge Q) \vee (P \wedge R)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

(b) The truth table should be identical to that of $P \Leftrightarrow Q$. This replacement rule makes sense since the only combinations of truth values where $P \Leftrightarrow Q$ is true are those where P and Q have the same truth value.

[1]FYI, the rules discussed here are called *distribution*, *material equivalence*, *material implication*, and *exportation*, respectively.

(c) The first rule is correct, and the second rule has the error in it. This can be checked by using a truth table to find the truth values of both sides of the replacement rules over all possibilities of the truth values of the logical statements; if the truth values of the two sides of the replacement rule don't match up then there has to be something wrong. For the second rule, we get that

P	**Q**	**R**	$(P \wedge Q) \Rightarrow R$	$P \Rightarrow (Q \vee R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

and it is easy to see that the truth values of $(P \wedge Q) \Rightarrow R$ and $P \Rightarrow (Q \vee R)$ do not match up over all possible truth values of P, Q, R, so the two sides cannot be equivalent.

For the bonus challenge problem, this replacement rule should actually read

$$(P \wedge Q) \Rightarrow R \quad \rightsquigarrow \quad P \Rightarrow (Q \Rightarrow R).$$

(W3) Calculus is about limits. However, it is likely that you were not given very precise definition of this notion. In this problem, we look at mathematically quantified definitions of a limit.

(a) Let $(x_n) = x_1, x_2, \ldots$ be a sequence of real numbers. We call x the *limit* of the sequence (x_n) if the following condition holds:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall n > N, \quad |x_n - x| < \epsilon.$$

Parse this statement in English and give some kind of intuitive explanation as to what this definition means.

(b) Using quantifiers, give the negation of the above statement.

(c) This definition of a limit can be expanded to define the limit of a function: Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the precise mathematical meaning of $\lim_{x \to c} f(x) = L$ is

$$\forall \epsilon > 0, \quad \exists \delta > 0 \quad \text{s.t.} \quad |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

Can you write down a similar expression for $\lim_{x \to \infty} f(x) = L$?

(d) Use quantifiers to express Fermat's Last Theorem:
No integers a, b, c satisfy the equation $a^n + b^n = c^n$ for any integer exponent n greater than 2.

Answer. (a) What this statement means is that we can say that a sequence (x_n) of real numbers tends towards a real number x if for any positive number ϵ, we can ensure that all values of the sequence (x_n) stay within the interval $(x - \epsilon, x + \epsilon)$ beyond a certain point in the sequence (namely, x_N).

(b) $\exists \epsilon > 0 \quad \text{s.t.} \quad \forall N \in \mathbb{N}, \quad \exists n > N \quad \text{so} \quad |x_n - x| \geq \epsilon.$

(c) $\forall \epsilon > 0, \quad \exists M \in \mathbb{R} \quad \text{s.t.} \quad \forall x > M, \quad |f(x) - L| < \epsilon.$

(d) $\forall n \in \mathbb{N} \quad \text{s.t.} \quad n > 2, \quad \nexists a, b, c \in \mathbb{Z} \quad \text{s.t.} \quad a^n + b^n = c^n.$

(W4) In class, you saw the Sheffer stroke, a logical operator ("NAND") defined as follows:

P	**Q**	$\mathbf{P \uparrow Q}$
T	T	F
T	F	T
F	T	T
F	F	T

Recall that $P \uparrow P = \sim P$. Work out expressions for both $P \wedge Q$ and $P \vee Q$ using only $P, Q, \uparrow$, and parentheses.

Answer. One can verify that

$$P \wedge Q \quad \rightsquigarrow \quad (P \uparrow Q) \uparrow (P \uparrow Q)$$

and

$$P \vee Q \quad \rightsquigarrow \quad (P \uparrow P) \uparrow (Q \uparrow Q).$$

(W5) (a) Do you have any remaining questions about the concepts and topics covered in class so far?
(b) How long did this worksheet take you to complete?
(c) Give a difficulty rating of this worksheet from 1 to 5, where 1 is "too easy", 3 is "perfect", and 5 is "too difficult".

Heads-up!

Last week's (W2) had botched indexing notation, though it was probably possible to guess the intention of the problem. Here it is again with corrected notation. Compare the two, but you don't have to re-do the problem!

For each $n \in \mathbb{N}$, let $A_n := (\frac{-1}{n}, \frac{1}{n})$, $B_n := [n, n + \frac{1}{n})$, and $C_n := \{x \in \mathbb{R} \mid n < x, x^2 \in \mathbb{Q}\}$. Now consider the collections $\mathcal{A} = \{A_n\}$, $\mathcal{B} = \{B_n\}$, $\mathcal{C} = \{C_n\}$, each indexed over $\mathbb{N}$.

(1) Which sets within $\mathcal{A}, \mathcal{B}, \mathcal{C}$ does the real number 5.1 belong to?
(2) Find the union and intersection of the sets in $\mathcal{A}$. Is $\mathcal{A}$ a nested collection?
(3) Let m be a positive integer. Show that $B_m \cap C_m \neq \emptyset$.