MATH 61-02: WORKSHEET 12 (§7.2)

(W1) Consider the Fig 7.7 in the book (p253), which shows airline connections between cities.

(a) Ordering the cities alphabetically, write the adjacency matrix A for the graph.

(b) Show by hand that it is possible to get from Seattle to Miami in 10 flights.

(c) Using a computer (or by hand if you want), compute A^8 and A^{10}. How many ways are there to get from Seattle to Miami in 10 flights? How about Chicago to Atlanta in 8 flights? Explain the connection.

Answer. (a) The adjacency matrix A is as follows:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Here's one sample schedule: SEA-CHI, CHI-NYC, NYC-ATL, ATL-CHI, CHI-ATL, ATL-CHI, CHI-ATL, ATL-CHI, CHI-ATL, ATL-MIA. (I wouldn't want to fly this route, though!)

(c) We have that A^8 is the following matrix:

$$\begin{pmatrix} 2069 & 2324 & 1790 & 1382 & 606 & 1378 & 772 \\ 2324 & 2847 & 2150 & 1546 & 772 & 1622 & 850 \\ 1790 & 2150 & 1709 & 1216 & 610 & 1306 & 696 \\ 1382 & 1546 & 1216 & 931 & 412 & 940 & 528 \\ 606 & 772 & 610 & 412 & 226 & 461 & 235 \\ 1378 & 1622 & 1306 & 940 & 461 & 1004 & 543 \\ 772 & 850 & 696 & 528 & 235 & 543 & 308 \end{pmatrix}$$

and that A^{10} is the following matrix:

$$\begin{pmatrix} 19628 & 22704 & 17603 & 13166 & 6098 & 13489 & 7391 \\ 22704 & 27123 & 20880 & 15209 & 7391 & 15883 & 8492 \\ 17603 & 20880 & 16351 & 11873 & 5775 & 12492 & 6717 \\ 13166 & 15209 & 11873 & 8856 & 4114 & 9111 & 4997 \\ 6098 & 7391 & 5775 & 4114 & 2069 & 4393 & 2324 \\ 13489 & 15883 & 12492 & 9111 & 4393 & 9564 & 5171 \\ 7391 & 8492 & 6717 & 4997 & 2324 & 5171 & 2847 \end{pmatrix}$$

Noting that Chicago is represented in the second row of the matrix and Atlanta is represented in the first row of the matrix, the number of ways to get from Chicago to Atlanta in 8 flights is the number in the second row and first column of A^8 - namely, 2324. Similarly, noting the Seattle is represented in the last row of the matrix and Miami is represented in the fifth row of the matrix, the number of ways to get from Seattle to Miami in 10 flights is also 2324.

Why are these two numbers the same? Well, for any path of length 10 in this graph from Seattle to Miami, it's clear that the first edge must go from Seattle to Chicago and the last edge must go from Atlanta to Miami. But then this forms a natural bijection between walks of length ten from Seattle to Miami and walks of length eight from Chicago to Atlanta - for any walk of length ten from Seattle to Miami, we can remove the first and final edge to get a walk of length eight from Chicago to Atlanta, and for any walk of length eight from Chicago to Atlanta, we can add an edge before the first step of the walk from Seattle to Chicago and an edge after the last step of the walk from Atlanta to Miami to get a walk of length ten from Seattle to Miami.

(W2) The second midterm included a question about the relation $R = \{(x,x),(x,y),(y,y),(y,x),(z,z)\}$ on $S = \{x,y,z\}$ where you were supposed to check that it is transitive. Draw the graph associated to the relation, and use the adjacency matrix to verify transitivity.

Answer. The graph associated to this relation is a digraph consisting of three vertices x, y, z with directed loops at every vertex and directed edges going from x to y and y to x.

The adjacency matrix A of R is as follows:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can compute that the matrix A^2 is then as follows:

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The transitivity test (Theorem 7.2.7) says that if the A^2 matrix has any nonzero entries, then the A matrix must also be nonzero in the corresponding places. There are six such positions, and the test succeeds, so we've verified transitivity.

Quick reminder of why this test works: those six nonzero entries of A^2 are telling you about paths of length 2 between particular vertices. For instance $(A^2)_{1,2} = 2$ tells you that there are two paths from v_1 to v_2 of length two. But such a path tells us that there's some (v_1, b) and also (b, v_2) in the data of the relation, and transitivity recognizes this as a "chain" and requires that (v_1, v_2) is also present.