MATH 61-02: WORKSHEET 11 (§7.1)

(W1) (a) For a vertex v in a graph $G = (V, E)$, let $\deg(v)$ be the *degree* of v, which is the number of times v appears as the endpoint of an edge. (So loops count double.) What is the smallest possible graph with at least one loop where every vertex has odd degree?

(b) Let K_n be the complete graph on n vertices (the simple graph where each vertex is connected by one edge to each other vertex). Sketch K_4, K_5, and K_6. For general n, what is $|E(K_n)|$ and what is the degree of each vertex?

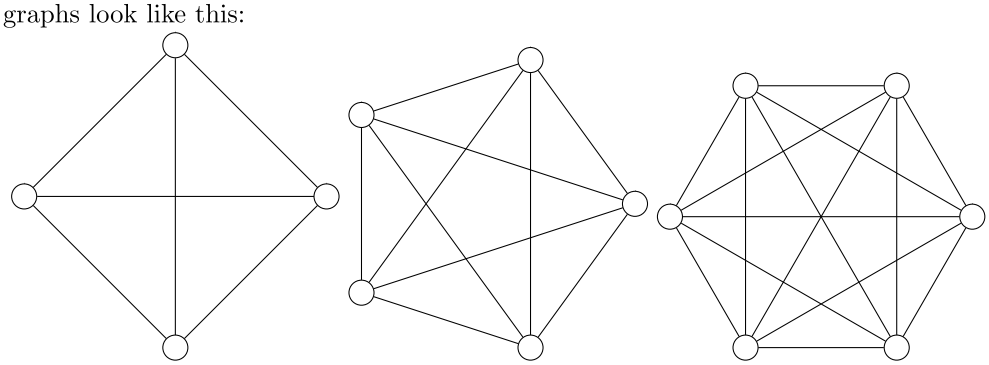
(c) Prove that $\sum_{v \in V} \deg(v) = 2|E|$ for any graph.

(d) Is it possible for a graph to have 11 vertices, all of which have degree 3?

(e) Is it possible for a graph to have 19 vertices, each of which have degree 1, 5, or 9? (Hint: consider the degree sum mod 4.)

Answer. (a) A graph on one vertex v with any number of loops must have $\deg(v)$ even, so our graph must have at least two vertices. It is easy to see that if we have two vertices v_0, v_1 with a loop at v_0 and an edge from v_0 to v_1, this satisfies the given constraint.

(b) The graphs look like this:



In general, $|E(K_n)| = \binom{n}{2}$, since an edge is uniquely defined by two distinct vertices and so it suffices to count all pairs of points in K_n. For each vertex $v \in K_n$, $\deg(v) = n - 1$, since we connect v to every point aside from itself.

(c) Suppose for a graph G we take the sum $\sum_{v \in V} \deg(v)$. This adds up the degrees of all vertices, which counts the edge-ends. Since every edge has two of those, this double-counts the edges, so adds up to $2|E|$.

(d) No: Suppose it did. Then we would have that $\sum \deg(v) = 11 \cdot 3 = 33 = 2|E|$, but it doesn't make sense to have $16\frac{1}{2}$ edges.

(e) No: Suppose it did, and let a, b, and c be the number of vertices with degree $1, 5, 9$, respectively. Then we would have that $\sum \deg(v) = 1a + 5b + 9c$, for some a, b, c satisfying $a + b + c = 19$. But let's consider everything mod 4. Since $5 \equiv 1 \pmod 4$ and $9 \equiv 1 \pmod 4$, we can reduce and obtain $a + 5b + 9c \equiv a + b + c$. Now we see that the degree sum has the same remainder as $19 \equiv 1 \pmod 4$, and that's impossible because it has to be even.

Actually, it's nice to practice your modular arithmetic and all, but there's a much simpler way to think about this: if all the degrees are odd, and you add up 19 odd numbers, you'll get an odd total. But we know that the total should be even!

(f) Suppose G is a simple graph (no loops) with $|V(G)| = n$. Show that if the degree of every vertex in G is at least $\frac{n-1}{2}$, then G is connected. (First convince yourself this is true for $n = 2, 3, 4$.)

There are many different ways to do this problem. Here are four of them:

Proof 1. Suppose G were not connected. Then it must have at least two connected components. In each of these connected components, there must be a vertex v with degree at least $\frac{n-1}{2}$, so each of these connected components must have at least $1 + \frac{n-1}{2}$ vertices (counting v as well). But since we have at least two of these connected components, that implies that G must have at least $2 + (n-1) = n+1$ vertices. Contradiction!

Proof 2. Suppose G were not connected. Then it must have at least two connected components, so there exists a partition of $V(G)$ into two sets of $1 \leq k \leq n-1$ and $n-k$ vertices with no edges between the two sets. But since G is simple, the first set of vertices can have at most $\binom{k}{2}$ edges among them, and the second set of vertices can have at most $\binom{n-k}{2}$ edges among them. It is easy to see that this sum $\binom{k}{2} + \binom{n-k}{2}$ is maximized when $k = \lfloor \frac{n}{2} \rfloor$. Then, the resulting sum will be at most $(\frac{n}{2})(\frac{n}{2} - 1) = \frac{n^2}{4} - \frac{n}{2}$, but the degree condition we have on our vertices combined with part (c) of the last question gives us that we have at least $\frac{n^2}{4} - \frac{n}{4}$ edges, a contradiction.

Proof 3. Take two vertices x and y in G. Now, either x and y are adjacent, or they are not. If they are not, since the degree of every vertex in G is at least $\frac{n-1}{2}$, the combined degrees of x and y must be at least $n-1$, and by the Pigeonhole Principle, there must exist a vertex v such that x and y are both adjacent to v. Hence any two vertices are either adjacent or have a common neighbor, and so the graph is clearly connected.

Proof 4. I make the same claim I showed in Proof 3, but go by contradiction: Every pair of distinct vertices are either adjacent or have a common neighbor. Suppose not. Then there exist two vertices u, v that aren't adjacent and don't have a common neighbor. So that means that all the neighbors of u and v must be distinct, so as the degrees of u, v are at least $\frac{n-1}{2}$ each, they have $n-1$ distinct neighbors, and adding u and v gives us that the graph has $n+1$ vertices. Contradiction!