MATH 61-02: WORKSHEET 1 (§1.1-1.2)

(W1) Let $\mathcal{P}(X)$ denote the power set of a set X. Let $A = \{3, 7\}$ and $B = \{4, 7, 12\}$.

(a) What is $\mathcal{P}(A \cup B)$? What is $\mathcal{P}(A) \cup \mathcal{P}(B)$? Name one element contained in one of these sets but not the other. Name one element contained in both of these sets.

(b) Let S and T be arbitrary sets. When does $\mathcal{P}(S \cup T) = \mathcal{P}(S) \cup \mathcal{P}(T)$? Explain your reasoning.

Answer. (a) $\mathcal{P}(A \cup B) = \{\emptyset, \{3\}, \{4\}, \{7\}, \{12\}, \{3, 4\}, \{3, 7\}, \{3, 12\}, \{4, 7\}, \{4, 12\}, \{7, 12\}, \{3, 4, 7\}, \{3, 4, 12\}, \{3, 7, 12\}, \{4, 7, 12\}, \{3, 4, 7, 12\}\}$, and $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{3\}, \{4\}, \{7\}, \{12\}, \{3, 7\}, \{4, 7\}, \{4, 12\}, \{7, 12\}, \{4, 7, 12\}\}$.

(b) In general, $\mathcal{P}(S \cup T) = \mathcal{P}(S) \cup \mathcal{P}(T)$ when $S \subseteq T$ (or $T \subseteq S$). Though a proof was not necessary on this assignment, here is a proof that this is a necessary and sufficient condition: Suppose that $S \subseteq T$. Then $\mathcal{P}(S) \subseteq \mathcal{P}(T)$, $S \cup T = T$ and $\mathcal{P}(S) cup \mathcal{P}(T) = \mathcal{P}(T)$, so $\mathcal{P}(S \cup T) = \mathcal{P}(T) = \mathcal{P}(S) \cup \mathcal{P}(T)$. On the other hand, suppose that it is not true that one of S and T is a subset of the other. Then there is an element $s \in S$ and $t \in T$ such that $s \notin T$ and $t \notin S$. But then the set $\{s, t\}$ is an element of $\mathcal{P}(S \cup T)$ but is not an element of $\mathcal{P}(S) \cup \mathcal{P}(T)$.

(W2) For each $n \in \mathbb{N}$, let $A_n := (\frac{-1}{n}, \frac{1}{n})$, $B_n := [n, n + \frac{1}{n})$, and $C_n := \{x \in \mathbb{R} \mid n < x, x^2 \in \mathbb{Q}\}$. Now consider the collections $\mathcal{A} = \{A_n\}$, $\mathcal{B} = \{B_n\}$, $\mathcal{C} = \{C_n\}$, each indexed over $\mathbb{N}$.

(a) Which sets within $\mathcal{A}, \mathcal{B}, \mathcal{C}$ does the real number 5.1 belong to?

(b) Find the union and intersection of the sets in $\mathcal{A}$. Is $\mathcal{A}$ a nested collection?

(c) Let m be a positive integer. Show that $B_m \cap C_m \neq \emptyset$.

Answer. Note that this question was botched - the indexing notation was unclear. A version with corrected notation is shown above (and is also provided on Worksheet 2), and the answers to the question as originally intended follow.

(a) 5.1 does not belong to any sets in $\mathcal{A}$. 5.1 belongs to $B_5 \in \mathcal{B}$. 5.1 belongs to $C_1, C_2, C_3, C_4, C_5 \in \mathcal{C}$.

(b) The union of the sets in $\mathcal{A}$ is the open interval $(-1, 1)$. The intersection of the sets in $\mathcal{A}$ is $\{0\}$. A is a nested collection.

(c) It is easily verified that for any m, the rational number $m + \frac{1}{2m}$ lies in both B_m and C_m, so $B_m \cap C_m \neq \emptyset$.

(W3) Let our universal set be $U = [6]$. Let $A = \{1, 2, 4, 5\}, B = \{1, 3, 5, 6\}, C = \{4, 5\}, D = \{1, 2, 6\}$, $E = \{2, 3, 6\}$. Use these sets and parentheses, unions, intersections, and complements to express the following sets. (It is possible to create all six.) As an example, the set $\{2, 6\}$ is equal to $D \cap E$, among other possible expressions.

(a) $\{1, 4, 5\}$

(b) $\{2, 4\}$

(c) $\{2\}$

(d) $\{3, 4\}$

(e) $\{1, 3, 6\}$

(f) $\{1, 3, 5\}$

Answer. Multiple answers are possible for each part, but here are some sample answers:

(a) E^c

(b) B^c

(c) $(D \setminus B) \cap E$

(d) $((B \cap E) \setminus D) \cup (C \setminus B)$

(e) $B \setminus C$

(f) $B \setminus (D \cap E)$

(W4) Consider the set $S_k := \{x = ak \mid a \in \mathbb{N}\}$ defined for $k \in \mathbb{N}$.

(a) Give a qualitative description of the set S_k for a given k.

(b) For what values of k is $12 \in S_k$? In general, suppose that $c \in \mathbb{N}$. For what values of k is $c \in S_k$?

(c) What is $S_2 \cap S_3$? What is $S_9 \cap S_{12}$? Can you find a way to describe the set $S_m \cap S_n$ for two positive integers m and n in general? (Can you extend this to more than two positive integers?)

(d) Can you find a set $T \subseteq \mathbb{N}$ such that

$$\bigcup_{t \in T} S_t = \mathbb{N} \setminus \{1\}?$$

Answer. (a) S_k is the set consisting of the multiples of k.

(b) $12 \in S_k$ for $k = 1, 2, 3, 4, 6, 12$. In general, $c \in S_k$ when c is a factor of k.

(c) $S_2 \cap S_3 = S_6$ and $S_9 \cap S_{12} = S_36$. In general, $S_m \cap S_n = S_{lcm(m,n)}$, where $lcm(m, n)$ is the least common multiple of m and n.

(d) Multiple answers were possible. The answer we had was $T = \{primenumbers\}$, but $T = \mathbb{N} \setminus \{1\}$ works as well.

(W5) (a) Do you have any remaining questions about the concepts and topics covered in class so far?

(b) Do you feel like this worksheet was too easy, too difficult, too long, or too short (if you are considering the "worksheet track")? Any suggestions?