Mathematics 51 | **Differential Equations**
Examination III | **May 6, 2013, 8:30–10:30am**

Tufts UNIVERSITY

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. *Please box your answers and cross out any work you do not want graded.* Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (10 points)

a. Show that $x(t) = t+4$ is a solution of the differential equation $\sin(t)D^3x + 4Dx - x = -t$.

Solution: $\sin(t)D^3x + 4Dx - x = 0 + 4\cdot 1 - (t+4) = -t$.

b. Write $\sin(t)D^3x + 4Dx - x = -t$ as a system of differential equations.

Solution: Write $x_1 = x$, $x_2 = x'$, $x_3 = x''$ to get

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= \frac{1}{\sin t}x_1 - 4\frac{1}{\sin t}x_2 - \frac{t}{\sin t} \end{aligned} \quad \text{or} \quad D\vec{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dfrac{1}{\sin t} & -\dfrac{4}{\sin t} & 0 \end{pmatrix}\vec{x} - \begin{pmatrix} 0 \\ 0 \\ \dfrac{t}{\sin t} \end{pmatrix}$$

c. Give *one* solution to the system *in vector form.*

Solution: $x_1 = x = t+4$, $x_2 = x' = 1$, $x_3 = x'' = 0$ gives $\vec{x}(t) = \begin{pmatrix} t+4 \\ 1 \\ 0 \end{pmatrix}$.

2. (10 points) Solve the initial-value problem $\dfrac{dx}{dt} = x^2, \quad x(0) = -3$.

Solution: $-\dfrac{1}{x} = \displaystyle\int \frac{dx}{x^2} = \int dt = t + C$, so $C = 1/3$ and $x(t) = -\dfrac{1}{t+1/3}$.

3. (10 points) Find all solutions (*in vector form*) to the equation $D\vec{x} = \begin{pmatrix} 5 & 3 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}\vec{x}$.

Solution: $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are eigenvectors for 1, 7, respectively, and $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ are generalized eigenvectors for the double eigenvalue 2 (obtained by solving $0 = \det\begin{pmatrix} 5-\lambda & 3 \\ -3 & -1-\lambda \end{pmatrix}$).

$$\vec{x}(t) = c_1e^t\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2e^{7t}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_3e^{2t}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 3t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\right] + c_4e^{2t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\right].$$

4. (10 points) Find all solutions to the equation $D\vec{x} = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ e^t \end{pmatrix}$

Solution: By inspection, the general solution of the associated homogeneous differential equation is $\vec{h}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. (Without inspection, do the usual: find the eigenvalues 3 (with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$) and 5 (with eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$) to get this general solution.)

Variation of parameters: $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 1 \\ e^t \end{pmatrix}$ gives $c_1 = -e^{-3t}/3$, $c_2 = -e^{-4t}/4$ and $\vec{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/3 \\ e^t/4 \end{pmatrix}$.

5. (10 points) *Use the Laplace transform* to solve the initial value problem $D^2x + Dx - 2x = 2$ with $x(0) = -1$ and $x'(0) = 0$. No credit by any other method.

Solution: $(s^2 + s - 2)\mathscr{L}[x] + s + 1 = \frac{2}{s}$, so $\mathscr{L}[x] = \frac{-s^2 - s + 2}{s(s^2 + s - 2)} = -\frac{1}{s}$ and $x(t) = -1$ for all t.

6. (10 points) Find all solutions to the differential equation $((D-1)^2 + 1)(D+3)Dx = 12$.

Solution: The general solution of the associated homogeneous differential equation is $h(t) = c_1 e^t \cos t + c_2 e^t \sin t + c_3 e^{-3t} + c_4$; a simplified guess for a particular solution is kt; plug this in to get $6k = 12$, so the general solution is $x(t) = c_1 e^t \cos t + c_2 e^t \sin t + c_3 e^{-3t} + c_4 t + 2t$.

7. (20 points) Consider the system $\frac{dx}{dt} = -x - y^2 \qquad \frac{dy}{dt} = y(2 - x)$.

a. Is the function $E(x, y) = x^2 - y^2$ a constant of motion?

Solution: $2x(-x - y^2) - 2y \cdot y(2 - x) = -2x^2 - 4y^2 < 0$ except when $(x, y) = (0, 0)$, so no.

b. Is the function $E(x, y) = x^2 - y^2$ a Lyapunov function?

Solution: Yes, by the computation in **a.** it (strictly!) decreases along nonconstant integral curves.

c. Find all equilibria of this system.

Solution: $0 = x' = -x - y^2$ and $0 = y' = y(2 + y^2)$ give $y = 0$, $\quad x = -y^2 = 0$. So $(0, 0)$ is it.

d. Find the linearization at each equilibrium and decide whether its phase portrait matches any of those at the end of the examination sheet; if so, identify which of these pictures it matches.

Solution: Dropping higher-order terms, the linearization at $(0, 0)$ is $\frac{dx}{dt} = -x$, $\frac{dy}{dt} = 2y$. (Or plug $(x, y) = (0, 0)$ into $A_{(x,y)} = \begin{pmatrix} \frac{\partial}{\partial x}(-x - y^2) & \frac{\partial}{\partial y}(-x - y^2) \\ \frac{\partial}{\partial x}(y(2 - x)) & \frac{\partial}{\partial y}(y(2 - x)) \end{pmatrix} = \begin{pmatrix} -1 & -2y \\ -y & 2 - x \end{pmatrix}$.) $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ has eigenvalues -1 (eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$) and 2 (eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$); this matches **V.**

e. For each equilibrium decide whether the Hartman–Grobman Theorem applies.

Solution: Yes, the eigenvalues of the linearization matrix have real parts -1 and 2; both nonzero.

f. Classify each equilibrium as stable or unstable.

Solution: Unstable by parts **d.** and **e.**

g. Classify each equilibrium as attractor, repeller or neither.

Solution: Neither, by parts **d.** and **e.**

h. Are there closed integral curves for this system?

Solution: No, for several reasons: • A closed integral curve must go around an equilibrium, and the only equilibrium here is a saddle, which makes this impossible. • Since $y' = y(2-x)$, solutions can not cross the x-axis, so a closed integral curve would have to lie in the upper half-plane or in the lower half-plane, neither of which contains an equilibrium around which to go. • When $x = 0$ then $x' = -y^2 \leq 0$, so integral curves cannot cross from the left half-plane to the right half-plane, so a closed integral curve would have to lie in the left half-plane or in the right half-plane, neither of which contains an equilibrium around which to go. • There is a Lyapunov function (which strictly decreases along nonconstant solutions; see **a.**, **b.**!).

8. (10 points) Consider the system $\dfrac{dx}{dt} = x^2 - y^2 \qquad \dfrac{dy}{dt} = yx - y.$

a. Find all equilibrium points.

Solution: $0 = \dfrac{dy}{dt} = yx - y = y(x-1)$ implies $y = 0$ or $x = 1$. If $0 = \dfrac{dx}{dt} = x^2 - y^2$, then $x^2 = y^2$, so $y = 0$ implies $x = 0$ and $x = 1$ implies $y = \pm 1$. So they are $(0,0)$, $(1,1)$ and $(1,-1)$.

b. For each equilibrium decide whether the phase portrait of the linearization matches any of those at the end of the examination sheet; if so, identify which of these pictures it matches.

Solution: The linearization matrix at (x,y) is $A_{(x,y)} = \begin{pmatrix} 2x & -2y \\ y & x-1 \end{pmatrix}$, so

$A_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$; this brings about the general solution $\begin{pmatrix} c_1 \\ c_2 e^{-t} \end{pmatrix}$, as shown in **IV.**

$A_{(1,\pm 1)} = \begin{pmatrix} 2 & \mp 2 \\ \pm 1 & 0 \end{pmatrix}$ has eigenvalues $1 \pm i$, hence outward spirals. Take $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as a test point to see that at $(1,1)$ they go counterclockwise, as in **III.** and at $(1,-1)$ they go clockwise, as in **II.**

c. For each equilibrium decide whether the Hartman–Grobman Theorem applies.

Solution: $(0,0)$: No because 0 is an eigenvalue. $(1,1)$: Yes—all eigenvalues have real part $1 \neq 0$.

9. (10 points)

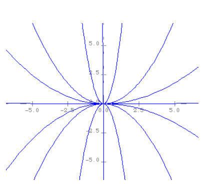
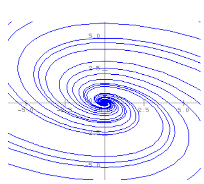
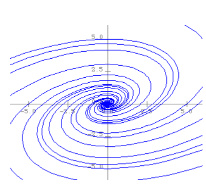
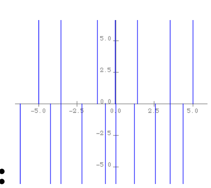
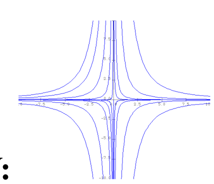
a. Find a recursion formula for the coefficients in the power series (centered at 0) for the solution of $D^2x - tDx + x = 0$ with $x(0) = 0$, $x'(0) = 1$.

Solution: Plug in $x(t) = \sum_{k=0}^{\infty} b_k t^k$ to get $\sum_{k=0}^{\infty} (k+2)(k+1) b_{k+2} t^k - b_k \cdot k t^k + b_k t^k = 0$ and $b_{k+2} = \dfrac{k-1}{(k+2)(k+1)} b_k$.

b. Find the power series.

Solution: The initial values give $b_0 = 0$, $b_1 = 1$, so the recursion gives $b_k = 0$ for all even k, $b_3 = 0$, and therefore $b_k = 0$ for all odd k thereafter. So $x(t) = \sum_{k=0}^{\infty} b_k t^k = t$. (Which checks!)

Phase portraits for matching up:

I: 
II: 
III: 
IV: 
V: 

END OF EXAMINATION