Mathematics 51 — **Tufts UNIVERSITY** — **Differential Equations**

Examination IV — **Dec. 14, 2012, 8:30–10:30am**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. *Please box your answers and cross out any work you do not want graded.* Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (5 points, no partial credit) Compute $x(2)$, where $x(t)$ is the solution of $tx' = 2x$ with $x(1) = 3$.

Solution: By inspection or separation of variables, $x(t) = 3t^2$, so $x(2) = 3 \cdot 2^2 = 12$.

2. (5 points) Determine whether

$$\begin{aligned} x_1(t) &= 3c_1 e^{4t} + c_2 e^{-4t} \\ x_2(t) &= c_1 e^{4t} + c_2 e^{-4t} \end{aligned}$$

describes the general solution of the system

$$\begin{aligned} x_1' &= 5x_1 - 3x_2, \\ x_2' &= 3x_1 - 5x_2. \end{aligned}$$

Solution: No, it is not even a solution unless $c_2 = 0$.

3. (5 points, no partial credit) The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ has a triple eigenvalue 1. Find 3 linearly independent generalized eigenvectors (you do not have to verify that they are linearly independent).

Solution: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ because there must be 3 linearly independent ones.

4. (10 points) Solve the initial-value problem $D\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Solution: Inspection gives eigenvalue 1 with eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and eigenvalue 0 (double) with eigenvectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, so the general solution is $c_1 e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The initial condition gives $c_1 = 1$, $c_2 = 1$, $c_3 = 0$, so the solution is $\vec{x}(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

5. (10 points) Find the general solution of $\quad x'' + x = \sec t$.

Solution: The general solution of the associated homogeneous equation is $c_1 \cos t + c_2 \sin t$. Variation of parameters: $\begin{aligned} c_1'(t) \cos t + c_2'(t) \sin t &= 0 \\ -c_1'(t) \sin t + c_2'(t) \cos t &= \sec t \end{aligned}$ gives $c_1'(t) = -\sin t \sec t = -\dfrac{\sin t}{\cos t}$ and $c_2'(t) = \cos t \sec t = 1$, so $c_1(t) = -\displaystyle\int \frac{\sin t}{\cos t}\, dt = \ln|\cos t|$ and $c_2(t) = t$. Thus, the general solution is $x(t) = c_1 \cos t + c_2 \sin t + \cos t \ln|\cos t| + t \sin t$.

6. (15 points, limited partial credit) Find the general solution of $D\vec{x} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix} \vec{x}$.

You may use without checking that $\begin{pmatrix} 1+\frac{t^2}{2} \\ t \\ 1 \\ t^2/2 \\ t \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} t \\ 1 \\ 0 \\ t \\ 1 \end{pmatrix}$ *are linearly independent solutions.*

Solution: Using the "free" eigenvalues 2, 3 with eigenvectors $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, respectively:

$$\vec{x}(t) = c_1 \begin{pmatrix} 1+(t^2/2) \\ t \\ 1 \\ t^2/2 \\ t \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} t \\ 1 \\ 0 \\ t \\ 1 \end{pmatrix} + c_4 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_5 e^{3t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

7. (10 points) Show that any set of vectors that includes $\vec{0}$ is linearly dependent.
(No credit for answers with over 15 words.)

Solution: $1 \cdot \vec{0} + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_n = \vec{0}$ is a *nontrivial* linear combination.

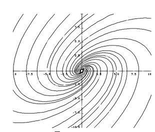
8. (15 points) Consider the system
$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = e^x y - x.$$

a. Find all equilibria.

Solution: $(0,0)$ only.

b. Draw the phase portrait of the *linearization* of each equilibrium.

Solution: Outward spirals, clockwise because $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.



c. For each equilibrium determine whether the Hartman–Grobman Theorem applies.

Solution: It does, since both eigenvalues have real part $1/2 \neq 0$.

d. Decide whether $E(x,y) = -x^2 - y^2$ is a constant of motion.

Solution: No: $(\partial E/\partial x)x' + (\partial E/\partial y)y' = -2xy - 2e^x y^2 + 2xy = -2e^x y^2 < 0$ for $y \neq 0$.

e. Decide whether $E(x,y) = -x^2 - y^2$ is a Lyapunov function.

Solution: Yes: $(\partial E/\partial x)x' + (\partial E/\partial y)y' = -2xy - 2e^x y^2 + 2xy = -2e^x y^2 < 0$ for $y \neq 0$.

f. Classify each equilibrium as an attractor, a repeller, or neither of these.

Solution: By inspection, $(0,0)$ is a global maximum for the Lyapunov function, hence a repeller.

Or: The linearization at the origin is $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, and it has eigenvalues $(1 \pm \sqrt{3}i)/2$, so $(0,0)$ is a repeller by the Hartman–Grobman Theorem.

g. Determine the stability of each equilibrium.

Solution: Being a repeller, the origin is unstable.

h. Decide whether this system of differential equations has a closed integral curve.

Solution: No: $\dfrac{\partial f}{\partial x}+\dfrac{\partial g}{\partial y}=e^x>0$ everywhere, **Or:** $2rr'=2xy+2y(e^xy-x)=2e^xy^2>0$ for $y\neq 0$, so $r=\sqrt{x^2+y^2}$ is strictly increasing, **Or:** There is a Lyapunov function.

9. (10 points) Solve $(D^3-D)x=\begin{cases}1 & t<2\\ 0 & t\geq 2\end{cases},\qquad x(0)=x'(0)=x''(0)=0.$

Solution: Rewrite as $(D^3-D)x=1-u_2(t)$ to get $\mathscr{L}[x]=\dfrac{1}{s^2(s^2-1)}-\dfrac{e^{-2s}}{s^2(s^2-1)}$.

$$\frac{1}{s^2(s^2-1)}=\frac{s^2-(s^2-1)}{s^2(s^2-1)}=\frac{1}{s^2-1}-\frac{1}{s^2}=\frac{1}{(s-1)(s+1)}-\frac{1}{s^2}=\frac{1}{2}\frac{(s+1)-(s-1)}{(s-1)(s+1)}-\frac{1}{s^2},$$

so $\mathscr{L}^{-1}[\dfrac{1}{s^2(s^2-1)}]=\dfrac{1}{2}(e^t-e^{-t})-t$, $\mathscr{L}^{-1}[\dfrac{e^{-2s}}{s^2(s^2-1)}]=u_2(t)\Big(\dfrac{1}{2}(e^{t-2}-e^{-(t-2)})-(t-2)\Big)$, and $x=\mathscr{L}^{-1}[\dfrac{1}{s^2(s^2-1)}-\dfrac{e^{-2s}}{s^2(s^2-1)}]=\dfrac{1}{2}(e^t-e^{-t})-t-\dfrac{1}{2}u_2(t)(e^{t-2}-e^{2-t}-2t+4)$.

10. (10 points) Consider the differential equation $x''+tx'+x=2t$.

a. Find the power-series expansion of the solution with $x(0)=0$, $x'(0)=1$.

Solution: $x(t)=t$ by inspection **Or:** write $x(t)=\sum_{k=0}^{\infty}b_kt^k$ to get $b_0=0$, $b_1=1$ from the initial values and $\displaystyle\sum_{k=0}^{\infty}b_{k+2}(k+2)(k+1)t^k+b_kkt^k+b_kt^k=2t$ from the differential equation. For $k=0$ this gives $2b_2+b_0=0$ hence $b_2=0$, and for $k=1$ we get $6b_3+2b_1=2$, hence $b_3=0$. For $k>1$, the recursion $b_{k+2}=-\dfrac{b_k}{k+2}$ gives $b_{k+2}=0$, so the expansion is $x(t)=t$.
[The expansion around $t=a$ would be $x(t)=a+(t-a)$.]
Equivalent approach: The initial data tell us that $x(t)=t+b_2t^2+\dots$, so write

$$\begin{aligned}x:&\qquad\qquad t+\ \ b_2t^2+\ b_3t^3+\\ tx':&\qquad\qquad t+\ 2b_2t^2+3b_3t^3+\\ x'':&2b_2+6b_3t+12b_4t^2+\end{aligned}$$

These add to $2t$ if we take all $b_i=0$.

b. Find the equivalent system of differential equations.

Solution: $D\vec{x}=\begin{pmatrix}0 & 1\\ -1 & -t\end{pmatrix}\vec{x}+\begin{pmatrix}0\\ 2t\end{pmatrix}$.

c. Find the solution of that system for which $\vec{x}(0)=\begin{pmatrix}0\\ 1\end{pmatrix}$.

Solution: $\vec{x}(t)=\begin{pmatrix}t\\ 1\end{pmatrix}$ from the first part.

11. (5 points, *no credit unless every answer is correct*) For each of the following vectors decide whether it is an eigenvector of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, and if so, provide the corresponding eigenvalue. For each part, your answer should be either "NO" or a number; please put all your answers on the inside front blue cover of your examination booklet.

a. $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ **Solution:** 2: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

b. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ **Solution:** 5: By inspection or a like computation.

c. $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ **Solution:** 0: By inspection. We now have 3 distinct eigenvalues and corresponding eigenvectors, so for each remaining part we only need to check whether the vector is a multiple of one of the 3 we already have; the answer is either "No" or the corresponding eigenvalue.

d. $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ **Solution:** 5

e. $\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ **Solution:** 0

f. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ **Solution:** 2 — and "No" to all remaining ones.

g. $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ **h.** $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ **i.** $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ **j.** $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ **k.** $\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ **l.** $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ **m.** $\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ **n.** $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

END OF EXAMINATION