Tufts
UNIVERSITY

Math 38 Spring 2011

Midterm Exam 3 with Solutions

1. (10 points): Check following sets of vectors for independence (NO PARTIAL CREDIT):

(a) $\begin{pmatrix}1\\7\\2\end{pmatrix}, \begin{pmatrix}2\\4\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\0\\-1\end{pmatrix}$

(b) $\begin{pmatrix}4\\3\\2\\1\end{pmatrix}, \begin{pmatrix}-2\\2\\4\\8\end{pmatrix}, \begin{pmatrix}6\\8\\8\\10\end{pmatrix}$

Solution:

(a) Not linearly independent.

(b) $2 \cdot \begin{pmatrix}4\\3\\2\\1\end{pmatrix} + \begin{pmatrix}-2\\2\\4\\8\end{pmatrix} - \begin{pmatrix}6\\8\\8\\10\end{pmatrix} = \vec{0}$. Hence, these vectors are not linearly independent.

2. (15 points): (a) Find the eigenvalues of the matrix $A = \begin{pmatrix}1 & 0 & 0\\-1 & 2 & 0\\0 & 1 & 3\end{pmatrix}$

(b) Find eigenvectors for each eigenvalue of the matrix A.

Solution:

(a) Eigenvalues are 1, 2, 3 since the matrix is lower triangular.

(b) By inspection, the eigenvector corresponding to $\lambda_3 = 3$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The eigenvector corresponding to $\lambda_2 = 2$ is obtained by solving the system $\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} v = 0$, so we get $v = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. The eigenvector corresponding to $\lambda_1 = 1$ is obtained by solving the system $\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} v = 0$, and we have $v = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$.

3. (10 points): The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has eigenvectors $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Find the corresponding eigenvalues.

Solution: For v one has $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and therefore, the eigenvalue is 2.

Similarly for w we have $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, and the eigenvalue is -1.

4. (10 points): Use the row reduction method to solve the following system:

$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 - x_5 &= 2 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 - 2x_5 &= 1 \\ -x_1 \qquad\quad - x_3 + 2x_4 + x_5 &= 1. \end{aligned}$$

No credit for any other method.

Solution: In matrix notation

$$\begin{pmatrix} 1 & 2 & 1 & -1 & -1 & | & -1 \\ 2 & 2 & 2 & -3 & -2 & | & 1 \\ -1 & 0 & -1 & 2 & 1 & | & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1, R_3 \to R_3 + R_1} \begin{pmatrix} 1 & 2 & 1 & -1 & -1 & | & -1 \\ 0 & -2 & 0 & -1 & 0 & | & -3 \\ 0 & 2 & 0 & 1 & 0 & | & 3 \end{pmatrix}$$

$$\xrightarrow{R_2 \to (-\frac{1}{2})R_2} \begin{pmatrix} 1 & 2 & 1 & -1 & -1 & | & -1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & | & \frac{3}{2} \\ 0 & 2 & 0 & 1 & 0 & | & 3 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 2 & 1 & -1 & -1 & | & -1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & | & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Let $x_3 = a, x_4 = b, x_5 = c$, then $x_1 = -a + 2b + c - 1, x_2 = -\frac{b}{2} + \frac{3}{2}$, so

$$x = \begin{pmatrix} -a + 2b + c - 1 \\ -\frac{b}{2} + \frac{3}{2} \\ a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ \frac{3}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

5. (20 points): Solve the system of linear differential equations $D\vec{x} = A\vec{x}$, where

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

Solution: Notice the block form. This is a decoupled pair of second order systems of differential equations. The characteristic polynomial of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is $(\lambda - 1)(\lambda - 3)$, so the eigenvalues are 1 and 3, 1 has eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, 3 has eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Next, $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ has eigenvalues $\pm 2i$. Take $2i$ and $\begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} v = 0$, so we get eigenvector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

$$e^{2it} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \cos 2t + i \sin 2t \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} sin2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix}.$$

So, the general solution is

$$c_1 e^t \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^3 t \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ \sin 2t \\ \cos 2t \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ -\cos 2t \\ \sin 2t \end{pmatrix}$$

6. (15 points): The matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

has a triple eigenvalue $\lambda = 1$ (You do not have to verify this). Find the generalized eigenvectors of A.

Solution: $(A - \lambda I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, the generalized eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

7. (20 points): Solve the following initial value problem:

$$D\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 10 \\ 20 \\ 10 \end{pmatrix}$$

Solution: The characteristic polynomial is $\lambda^3 - 2\lambda^2$, so $\lambda_1 = 2$, which gives eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and a double root $\lambda_2 = 0$. For $\lambda_2 = 0$, the corresponding eigenvectors, by inspection are $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. (You will get the same vectors if you solve the system $(A - \lambda I)^2 v = 0$).

So, the general solution is $x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Plugging in the initial values, we have $c_1 = 15, c_2 = 5, c_3 = -5$. Thus the solution is

$$x(t) = 15 \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} + 5 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + (-5) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$