Mathematics 51 **Tufts UNIVERSITY** **Differential Equations**
Examination III **December 3, 2012, 12–1:20pm**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (15 points) Find the general solution of $\begin{cases} x' = 2y \\ y' = 3y - x \end{cases}$.

Solution: The characteristic polynomial $-\lambda(3-\lambda)+2 = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$ of the coefficient matrix $\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$ gives eigenvalues 1 and 2. $A - 1I = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$, giving an eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for $\lambda = 1$; likewise, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 2$, so the general solution is $\vec{x}(t) = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, or $\begin{cases} x(t) = 2c_1 e^t + c_2 e^{2t} \\ y(t) = c_1 e^t + c_2 e^{2t}. \end{cases}$

2. (15 points) Find the general solution of $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x}$. [Hint: $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector.]

Solution: $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1+i \\ 1-i \end{pmatrix} = (1+i) \begin{pmatrix} 1 \\ -i \end{pmatrix}$, so the eigenvector $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ corresponds to the eigenvalue $1+i$, and the corresponding (complex) solution is

$e^{(1+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^t(\cos t + i \sin t) \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^t \begin{pmatrix} \cos t + i \sin t \\ \sin t - i \cos t \end{pmatrix} = e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + i e^t \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$.

The general solution then is $\mathbf{x}(t) = c_1 e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$.

3. (10 points) The matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ has a double eigenvalue 2 with generalized eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the general solution of $D\vec{x} = A\vec{x}$.

Solution: Using $A - 2I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$,

$$\mathbf{x}(t) = c_1 e^{2t}(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix}) + c_2 e^{2t}(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix}).$$

4. (15 points) Find the general solution of $D\vec{x} = A\vec{x} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$, where $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$.

Hint: The general solution of $D\vec{x} = A\vec{x}$ is $c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

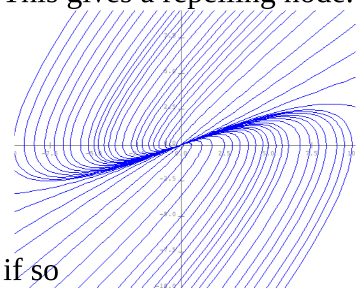
Solution: Thanks to the hint (or Problem 1.) we can do variation of parameters right away:

$$\left(\begin{array}{cc|c} 2e^t & e^{2t} & e^t \\ e^t & e^{2t} & e^t \end{array}\right) \to \left(\begin{array}{cc|c} e^t & 0 & 0 \\ e^t & e^{2t} & e^t \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ e^{-t} & 1 & e^{-t} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & e^{-t} \end{array}\right),$$

so $c_1' = 0$, $c_2' = e^{-t}$, $c_2 = -e^{-t}$; the general solution is $c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

5. (15 points) Sketch the phase portrait of the system $D\vec{x} = A\vec{x}$, where A is the matrix from problem **4.**

Solution: The hint in Problem 4 (or our solution of Problem 1.) tells us that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector for the eigenvalue 1 and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for the eigenvalue 2. This gives a repelling node.



6. (20 points) For the system of differential equations $\begin{aligned} \frac{dx}{dt} &= x - y \\ \frac{dy}{dt} &= -2x + x^2 \end{aligned}$

a. find all equilibria, and for each of them

b. determine whether the Hartman–Grobman Theorem applies to it, and if so

c. determine its stability,

d. decide whether it is an attractor, a repeller, or neither,

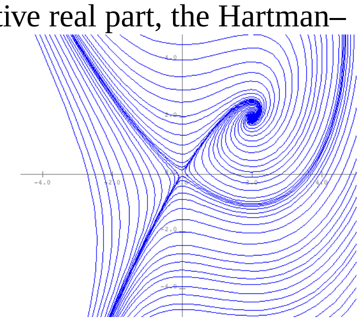
e. draw a phase portrait of the linearization.

Sketch the full phase portrait.

Solution: $\begin{cases} x - y = 0 \\ x(x-2) = 0 \end{cases}$ gives equilibria $(0,0)$ and $(2,2)$. The linearization is $\begin{pmatrix} 1 & -1 \\ 2x-2 & 0 \end{pmatrix}$.

At $(0,0)$ this becomes $\begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$, whose characteristic polynomial is $(1-\lambda)(-\lambda) - 2 = (\lambda+1)(\lambda-2)$. So the Hartman–Grobman Theorem applies, $(0,0)$ is a saddle, hence unstable but neither an attractor nor a repeller. $\lambda = -1 \leftrightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda = 2 \leftrightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

At $(2,2)$ the linearization matrix is $\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$ with characteristic polynomial $(1-\lambda)(-\lambda) + 2 = (\lambda - \frac{1}{2})^2 + \frac{7}{4}$, so the eigenvalues are $\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$. Since both have positive real part, the Hartman–Grobman Theorem applies, and $(2,2)$ is a repeller, hence unstable. (Spirals out, counterclockwise.)



7. (10 points) For the system $\begin{aligned} \frac{dx}{dt} &= -\sin x \cos y \\ \frac{dy}{dt} &= \cos x \sin y, \end{aligned}$

a. check whether $E(x,y) = \sin x \sin y$ is a constant of motion,

b. find one equilibrium (yes, just one—*but make sure you get it right: there will be no credit for any work on a point that is not an equilibrium, and no credit if you work on more than one point!*),

c. determine whether it is an attractor, a repeller, or neither. (*Give clear and complete reasoning!*)

Solution:

a. $\frac{dE}{dt} = -\cos x \sin y \sin x \cos y + \sin x \cos y \cos x \sin y = 0$—yes.

b. $(0,0)$.

c. The equilibrium is neither an attractor nor a repeller because there is a constant of motion (with isolated critical points) or by using the Hartman–Grobman Theorem in this particular case.

END OF EXAMINATION