**Mathematics 38** **Differential Equations**
**Exam III** **December 5, 2011**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

**1.** (10 points) Given the matrix $A = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$ and the eigenvector $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ find

**a.** the eigenvalue $\lambda$ of $A$ to which $\vec{v}$ corresponds and

**b.** the associated solution of $D\vec{x} = A\vec{x}$.

**Solution: a.** $\begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so $\lambda = -2$. **b.** $e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

**2.** (20 points) The $3 \times 3$ matrix $A = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ has a triple eigenvalue of $1$ (you do not have to verify this). Find the general solution of

$$D\vec{x} = A\vec{x}.$$

**Solution:** $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are (obviously!!) generalized eigenvectors. Each of these gives a solution $h_i(t) = e^t[I + t(A-I) + \frac{t^2}{2}(A-I)^2]\vec{v}_i =$

$$= e^t\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t\begin{pmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}\right]\vec{v}_i$$

$$= e^t\begin{pmatrix} 1+2t+t^2/2 & -3t-t^2 & t+t^2/2 \\ t+t^2/2 & 1-t-t^2 & t^2/2 \\ t^2/2 & t-t^2 & 1-t+t^2/2 \end{pmatrix}\vec{v}_i,$$

so the sought general solution is

$$\vec{x}(t) = c_1e^t\begin{pmatrix} 1+2t+t^2/2 \\ t+t^2/2 \\ t^2/2 \end{pmatrix} + c_2e^t\begin{pmatrix} -3t-t^2 \\ 1-t-t^2 \\ t-t^2 \end{pmatrix} + c_3e^t\begin{pmatrix} t+t^2/2 \\ t^2/2 \\ 1-t+t^2/2 \end{pmatrix}$$

**3.** (20 pts) Find the general solution of $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\vec{x}$. Hint: $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector.

**Solution:** $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1+i \\ * \end{pmatrix} = (1+i)\begin{pmatrix} 1 \\ * \end{pmatrix}$, so the eigenvector $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ corresponds to the eigenvalue $1+i$, and the corresponding (complex) solution is

$$e^{(1+i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} = e^t(\cos t + i\sin t)\begin{pmatrix} 1 \\ -i \end{pmatrix} = e^t\begin{pmatrix} \cos t + i\sin t \\ \sin t - i\cos t \end{pmatrix} = e^t\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + ie^t\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.$$

The general solution then is $\vec{x}(t) = c_1e^t\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2e^t\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$.

**4.** (20 pts) Find the general solution of $D\vec{x} = A\vec{x} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$, where $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$.

Hint: The general solution of $D\vec{x} = A\vec{x}$ is $c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

**Solution:** Thanks to the hint we can do variation of parameters right away:

$$\left(\begin{array}{cc|c} 2e^t & e^{2t} & e^t \\ e^t & e^{2t} & e^t \end{array}\right) \to \left(\begin{array}{cc|c} e^t & 0 & 0 \\ e^t & e^{2t} & e^t \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ e^{-t} & 1 & e^{-t} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & e^{-t} \end{array}\right),$$

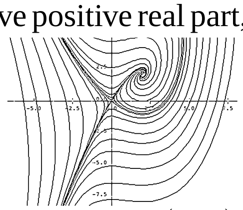
so $c_1' = 0$ and $c_2' = e^{-t}$, hence $c_2 = -e^{-t}$, so $c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the general solution.

**5.** (20 pts) For the system of differential equations
$$\frac{dx}{dt} = x - y$$
$$\frac{dy}{dt} = -2x + x^2$$

**a.** find the equilibria. (Make sure to get them right; otherwise there will be little partial credit.) For each equilibrium
**b.** state (with reason) whether the Hartman–Grobman Theorem applies,
**c.** determine its stability,
**d.** decide whether it is an attractor, a repeller, or neither.

**e.** Sketch a plausible global phase portrait.

**Solution:** $\begin{cases} x - y = 0 \\ x(x-2) = 0 \end{cases}$ gives equilibria $(0,0)$ and $(2,2)$. The linearization is $\begin{pmatrix} 1 & -1 \\ 2x-2 & 0 \end{pmatrix}$. At $(0,0)$ this matrix becomes $\begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$, whose characteristic polynomial is $(1-\lambda)(-\lambda) - 2 = (\lambda+1)(\lambda-2)$. So $(0,0)$ is a saddle, hence unstable but neither an attractor nor a repeller. The eigenvalue-eigenvector pairs are $-1$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and 2, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (no eigenvalues with zero real part, so the Hartman–Grobman Theorem applies). At $(2,2)$ the linearization matrix is $\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$ with characteristic polynomial $(1-\lambda)(-\lambda) + 2 = (\lambda - \frac{1}{2})^2 + \frac{7}{4}$, so the eigenvalues are $\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$. Since both have positive real part, the Hartman–Grobman Theorem applies, $(2,2)$ is a repeller, hence unstable.



**6.** (10 pts) Check whether $E(x,y) = \sin x \sin y$ is a constant of motion for
$$\frac{dx}{dt} = -\sin x \cos y$$
$$\frac{dy}{dt} = \cos x \sin y.$$

**Solution:** Yes: $\dfrac{dE}{dt} = -\cos x \sin y \sin x \cos y + \sin x \cos y \cos x \sin y = 0.$

*END OF EXAMINATION*

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