Math 38	Tufts University	Monday, March 12, 2012
Differential Equations	Department of Mathematics	12:00–1:20pm
	Exam 2	

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. A correct answer with no work may not necessarily score any points. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature, you are pledging that you have neither given nor received assistance on the exam. Any violations will be reported to the appropriate dean, and will result in an F for the course.
Please begin each answer on a new page.

1. (8 points) If $L = D^2 + 3D - 2$ find Le^{2t} and Lt^3e^{2t}.

2. (5 points) Use the definition of $\mathscr{L}x$ to find $\mathscr{L}(t+1)$.

3. (12 points) Find $\mathscr{L}f(t)$ for
   (a) $f(t) = \begin{cases} t^2 & t < 1 \\ t^2 - 1 & 1 \le t \end{cases}$
   (b) $f(t) = e^t \cos 3t$

4. (12 points) Find $\mathscr{L}^{-1}F(s)$ for
   (a) $F(s) = \dfrac{e^{-(s+1)}}{s+1}$
   (b) $F(s) = \dfrac{s+1}{s^2+3s+3}$

5. (6 points) Use the definition of convolution to compute $t * t$.

6. (12 points) Given
$$(N) \qquad (D-1)^2(D+1)x = 1$$
   (a) Find the equivalent 3×3 system (S_N).
   (b) Find the general solution of (N) and use it to find the general solution of (S_N).
   (c) Write (S_N) in matrix form.
   (d) Write the general solution of (S_N) in the form
$$c_1\vec{h_1} + c_2\vec{h_2} + c_3\vec{h_3} + \vec{p}$$

7. (10 points) Solve the initial value problem
$$(D-1)^2(D+2)x = 0, \qquad x(0) = x'(0) = 0, x''(0) = 9.$$

8. (10 points) Solve
$$(D^2+1)x = t + \sin t.$$

9. (10 points) Solve

$$(D^2 + 1)x = \tan t \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

(Hint: try variation of parameters)

10. (15 points) Solve the initial value problem

$$(D+1)^3 x = \begin{cases} e^{-t} & t < 3 \\ 0 & t \geq 3 \end{cases} \qquad x(0) = x'(0) = x''(0) = 0$$

Solutions to Exam 2 Math 38 12 March '12

1) $L = D^2 + 3D - 2 \qquad L' = L(D+2) = (D+2)^2 + 3(D+2) - 2 = D^2 + 7D + 8$

$$L e^{2t} = e^{2t} L' 1 = e^{2t} \cdot 8$$

$$L t^3 e^{2t} = e^{2t} L'(t^3) = (8t^3 + 21t^2 + 6t) e^{2t}$$

2) $\mathcal{L}(t+1) = \int_0^\infty e^{-st}(t+1)\,dt = \int_0^\infty t e^{-st}\,dt + \int_0^\infty e^{-st}\,dt$

$$= \lim_{h \to \infty} \left. -\frac{t e^{-st}}{-s} \right|_0^h - \left. \frac{1}{s^2} e^{-st} \right|_0^h - \left. \frac{1}{s} e^{-st} \right|_0^h. \quad \text{If } s > 0$$

$$\frac{-h e^{-sh}}{-s} \to 0 \qquad -\frac{1}{s^2} e^{-sh} \to 0 \text{ and } -\frac{1}{s} e^{-sh} \to 0 \text{ as } h \to \infty$$

and you are left with $+\frac{1}{s^2} + \frac{1}{s}$.

3) a) $f(t) = t^2 - u_1(t) \qquad \mathcal{L} f(t) = \frac{2}{s^3} - e^{-s} \mathcal{L} 1 = \frac{2}{s^3} - \frac{e^{-s}}{s}$

b) $f(t) = e^t \cos 3t = \frac{s-1}{(s-1)^2 + 9}$ (1st shift)

4) a) $\mathcal{L}^{-1} \frac{e^{-(s+1)}}{s+1} = e^{-1} \mathcal{L} \frac{e^{-s}}{s+1} = e^{-1} u_1(t) f(t-1)$ with

$f(t) = \mathcal{L}^{-1} \frac{1}{s+1} = e^{-t}$ so $e^{-1} u_1(t) e^{-(t-1)} = u_1(t) e^{-t}$

b) $\mathcal{L}^{-1} \frac{s+1}{s^2 + 3s + 3} = \mathcal{L}^{-1} \frac{s+1}{(s+\frac{3}{2})^2 + \frac{3}{4}} = e^{-\frac{3}{2}t} \mathcal{L}^{-1} \frac{s - 1/2}{s^2 + 3/4}$

$$= e^{-\frac{3}{2}t} \left[\cos \frac{\sqrt{3}}{2} t - \frac{1}{2\sqrt{3}/2} \sin \frac{\sqrt{3}}{2} t \right]$$

5) $\int_0^t u(t-u)\,du = t \int_0^t u\,du - \int_0^t u^2\,du = t \cdot \frac{t^2}{2} - \frac{t^3}{3} = \frac{t^3}{6}$

6) (N) $(D^3 - D^2 - D + 1)\, x = 1$ has gen. sol'n

$$c_1 e^t + c_2 t e^t + c_3 e^{-t} + 1$$

$p(t)=1$ is an obvious particular non-homog solution.

a)

$$D\vec{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = -x_1 + x_2 + x_3 + 1$$

b)

$$x_1 = c_1 e^t + c_2 t e^t + c_3 e^{-t} + 1$$
$$x_2 = c_1 e^t + c_2 (t+1) e^t - c_3 e^{-t}$$
$$x_3 = c_1 e^t + c_2 (t+2) e^t + c_3 e^{-t}$$

c) See a)

d)

$$\vec{x} = c_1 \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} t e^t \\ (t+1) e^t \\ (t+2) e^t \end{pmatrix} + c_3 \begin{pmatrix} e^{-t} \\ -e^{-t} \\ e^{-t} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

7) Using Laplace you get $(s-1)^2 (s+2)\, \mathcal{L}x - 9 = 0$

$$\mathcal{L}x = \frac{9}{(s-1)^2 (s+2)} = \frac{-1}{s-1} + \frac{3}{(s-1)^2} + \frac{1}{s+2}$$

or $x = -e^t + 3te^t + e^{-2t}$.

8 Annihilator of $t + \sin t$ is $D^2 (D^2+1)$

The new equation is $D^2 (D^2+1)^2 x = 0$

Simplified guess $p(t) = c_1 + c_2 t + c_3 t \sin t + c_4 t \cos t$

$(D^2+1)\, p(t) = c_1 + c_2 t + 2c_3 \cos t - 2c_4 \sin t = t + \sin t$

so $c_1 = 0$ $c_2 = 1$, $c_3 = 0$, $c_4 = -1/2$

particular solution $p(t) = t - \frac{1}{2} t \cos t$

general solution $\quad x = c_1 \cos t + c_2 \sin t + t - \frac{1}{2} t \cos t$

9) $(D^2+1)\,x = \tan t \qquad$ V.P. $\quad p = c_1(t)\cos t + c_2(t)\sin t$

$$c_1' \cos t + c_2' \sin t = 0$$

$$-c_1' \sin t + c_2' \cos t = \tan t$$

$$c_1' = \begin{vmatrix} 0 & \sin t \\ \tan t & \cos t \end{vmatrix} = -\sin t \tan t = \cos t - \sec t$$

$$c_2' = \begin{vmatrix} \cos t & 0 \\ -\sin t & \tan t \end{vmatrix} = \cos t \tan t = \sin t$$

$$c_1 = \sin t - \ln|\tan t + \sec t| \qquad c_2 = -\sin t$$

particular $\quad (\sin t - \ln|\sec t + \tan t|)\cos t - \sin t \cos t =$

$$= -\cos t \ln|\sec t + \tan t|$$

general $\quad c_1 \cos t + c_2 \sin t - \cos t \ln|\sec t + \tan t|.$

10) $(D+1)^3 x = e^{-t} - u_3(t)\, e^{-t} \qquad$ all init cond $= 0$ so

$$\mathcal{L}(D+1)^3 x = (s+1)^3 \mathcal{L}x = \frac{1}{s+1} - e^{-3s} e^{-3} \frac{1}{s+1}$$

$$\mathcal{L}x = \frac{1}{(s+1)^4} - e^{-3s} e^{-3} \frac{1}{(s+1)^4} = \cancel{\frac{t^3}{6} e^{-t} - u_3(t)\left[e^{-3} e^{-(t-3)} (t-3)^3\right]}$$

$$= \frac{t^3}{6} e^{-t} - u_3(t)\, e^{-3} \left(\frac{e^{-(t-3)} (t-3)^3}{6} \right)$$

$$= \frac{t^3}{6} e^{-t} - u_3(t) \frac{e^{-t}}{6} (t-3)^3$$