Midterm Exam 2 with Solutions

1. (15 points): Use the method of undetermined coefficients to find a particular solution of the equation

$$x'' + x' - 2x = -18te^{-2t}.$$

Solution: The characteristic equation corresponding to the homogeneous equation is $\lambda^2 + \lambda - 2 = 0$. The roots are $\lambda = 1$, $\lambda = -2$, and the general solution to the homogeneous equation is

$$H(t) = C_1 e^t + C_2 e^{-2t}.$$

The annihilator for the right side is $A(D) = (D+2)^2$. Therefore, any particular solution of the given equation is among the general solution to the equation $A(D)(D-1)(D+2)x = 0$, or $(D-1)(D+2)^3 x = 0$. The general solution of this equation is

$$x = C_1 e^t + (C_2 + C_3 t + C_4 t^2)e^{-2t}.$$

Hence, the simplified form for a particular solution has the form $x_p = (C_3 t + C_4 t^2)e^{-2t}$. Substituting this into equation one obtains $2C_4 - 3C_3 = 0$, and $-6C_4 = -18$, or $C_4 = 3$, $C_3 = 2$.

ANS: $\boldsymbol{x_p = (2t + 3t^2)e^{-2t}}$.

2. (10 points): Solve the following equation by the method of variation of parameters

$$2x'' + 2x = \sec t.$$

No credit by any other method.

Solution: The functions $h_1(t) = \cos t$ and $h_2(t) = \sin t$ generate the general solution to the homogeneous equation $2x'' + 2x = 0$. Hence, we look for a particular solution in the form

$$x_p = c_1(t)\cos t + c_2(t)\sin t.$$

The system

$$\cos(t)c_1' + \sin(t)c_2' = 0,$$
$$-\sin(t)c_1' + \cos(t)c_2' = \frac{1}{2}\sec t,$$

has a solution $c_1' = -\frac{1}{2}\tan t$ and $c_2' = \frac{1}{2}$, yielding $c_1(t) = \frac{1}{2}\ln|\cos t|$ and $c_2(t) = \frac{t}{2}$.

ANS: $\boldsymbol{x(t) = c_1\cos t + c_2\sin t + \frac{1}{2}\cos t\ln|\cos t| + \frac{t}{2}\sin t.}$

3. (10 points): Find the Laplace transform of the following functions:

(a) $te^{-t}\sin 2t$

(b) $\begin{cases} 2, & \text{if } t < 2, \\ t, & \text{if } 2 \le t < 3, \\ e^{t-3} + 3, & \text{if } 3 \le t. \end{cases}$

Solution:

(a) By the First Differentiation Formula, $\mathcal{L}[t\sin(2t)](s) = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right] = \frac{4s}{(s^2+4)^2}$.
Then, by the First Shift Formula, $\mathcal{L}[e^{-t}t\sin(2t)](s) = \frac{4(s+1)}{[(s+1)^2+4]^2}$.

(b) $\mathcal{L}\big[2 + u_2(t)\cdot(t-2) + u_3(t)\cdot[e^{t-3} + 3 - t]\big](s) = \frac{2}{s} + e^{-2s}\mathcal{L}[t](s) + e^{-3s}\mathcal{L}[e^t - t](s) = \frac{2}{s} + \frac{e^{-2s}}{s^2} + e^{-3s}(\frac{1}{s-1} - \frac{1}{s^2})$ (Second Shift Formula).

4. (10 points): Find the inverse Laplace transform of the following functions:

(a) $\dfrac{e^{-s}(1+s)}{2(s^2+1)}$

(b) $\dfrac{s}{s^2-2s+2}$

Solution:

(a) $\mathcal{L}^{-1}\left[\dfrac{e^{-s}(1+s)}{2(s^2+1)}\right](t) = \dfrac{1}{2}u_1(t)\left(\cos(t-1)+\sin(t-1)\right)$. (Second Shift Formula).

(b) $\mathcal{L}^{-1}\left[\dfrac{s}{s^2-2s+2}\right](t) = \mathcal{L}^{-1}\left[\dfrac{s-1}{(s-1)^2+1} + \dfrac{1}{(s-1)^2+1}\right](t) = e^t(\cos t + \sin t)$. (First Shift Formula).

5. (10 points): Use the following steps to compute the convolution $t * (\sin t - e^t)$:

(a) find the Laplace transform of $t * (\sin t - e^t)$;

(b) find a partial fraction decomposition of the Laplace transform obtained in Part (a);

(c) find the inverse Laplace transform of the partial fraction decomposition found in Part (b).

Solution:

(a) $\mathcal{L}[t * (\sin t - e^t)] = \mathcal{L}[t] \cdot \mathcal{L}[\sin t - e^t] = \frac{1}{s^2}\left(\frac{1}{1+s^2} - \frac{1}{s-1}\right)$

(b) $\frac{1}{s^2}\left(\frac{1}{1+s^2} - \frac{1}{s-1}\right) = \frac{1}{s^2(1+s^2)} - \frac{1}{s^2(s-1)} = \frac{1}{s^2} - \frac{1}{1+s^2} - \frac{s-(s-1)}{s^2(s-1)} = \frac{1}{s^2} - \frac{1}{1+s^2} - \frac{1}{s(s-1)} + \frac{1}{s^2}$
$= \frac{2}{s^2} - \frac{1}{1+s^2} - \frac{1}{s-1} + \frac{1}{s}$.

(c) $t * (\sin t - e^t) = \mathcal{L}^{-1}[\frac{2}{s^2} - \frac{1}{1+s^2} - \frac{1}{s-1} + \frac{1}{s}] = 2t - \sin t - e^t + 1$.

ANS: $\mathbf{2t - \sin t - e^t + 1}$.

6. (10 points): Compute $\mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)}\right]$ using the convolution formula. No credit by any other method.

Solution: $\mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) * \mathcal{L}^{-1}\left[\frac{1}{s-2}\right](t) = t * e^{2t} = \int_0^t ue^{2(t-u)}\,du$
$= e^{2t}\int_0^t ue^{-2u}\,du = e^{2t}(\frac{1}{4} - \frac{t}{2}e^{-2t} - \frac{1}{4}e^{-2t})$.

ANS: $\boldsymbol{\frac{e^{2t}}{4} - \frac{t}{2} - \frac{1}{4}}$**.**

7. (15 points): Solve the following initial value problem

$$x'' - 4x' + 5x = f(t),$$
$$x(0) = 0,\ x' = 1,$$

where $f(t) = \begin{cases} 5, & \text{if } t < 1 \\ 0, & \text{if } 1 \le t. \end{cases}$

Solution: Applying the Laplace transform to both sides of the equation we have

$$(s^2 - 4s + 5)\mathcal{L}[x] - 1 = \frac{5}{s} - e^{-s}\frac{5}{s},$$

so

$$\mathcal{L}[x] = \frac{1}{(s-2)^2+1} + \frac{5}{s[(s-2)^2+1]} - e^{-s}\frac{5}{s[(s-2)^2+1]}.$$

Now we have to compute inverse Laplace transforms of these three terms on the right. The first term has the inverse

$$\mathcal{L}^{-1}[\frac{1}{(s-2)^2+1}] = e^{2t}\sin t.$$

Second term can be written in the partial fraction form

$$\frac{5}{s[(s-2)^2+1]} = \frac{1}{s} - \frac{s-2}{(s-2)^2+1} + \frac{2}{(s-2)^2+1},$$

therefore,

$$\mathcal{L}^{-1}[\frac{5}{s[(s-2)^2+1]}] = 1 - e^{2t}\cos t + 2e^{2t}\sin t.$$

Inverse of the third term due to the Second Shift Formula is

$$\mathcal{L}^{-1}[e^{-s}\frac{5}{s[(s-2)^2+1]}] = u_1(t)\cdot[1 - e^{2(t-1)}\cos(t-1) + 2e^{2(t-1)}\sin(t-1)].$$

Hence,

$$\boldsymbol{x(t) = 1 - e^{2t}\cos t + 3e^{2t}\sin t - u_1(t)\cdot[1 - e^{2(t-1)}\cos(t-1) + 2e^{2(t-1)}\sin(t-1)].}$$

8. (10 points): Reduce the following higher order linear differential equations to equivalent systems (DO NOT SOLVE):

(a) $x'' + (e^t + t)x' = \sec t$

(b) $(D+1)^3 x = t$

Solution:

(a) Introducing $x_1 = x$ and $x_2 = x'$ we have

$$x_1' = x_2,$$
$$x_2' = -(e^t + t)x_2 + \sec t.$$

ANS: $Dx = Ax + E(t)$ with $A = \begin{pmatrix} 0 & 1 \\ 0 & -e^t - t \end{pmatrix}$ and $E(t) = \begin{pmatrix} 0 \\ \sec(t) \end{pmatrix}$

(b) Introducing $x_1 = x$, $x_2 = x'$ and $x_3 = x''$ we have

$$x_1' = x_2,$$
$$x_2' = x_3,$$
$$x_3' = -x_1 - 3x_2 - 3x_3 + t.$$

ANS: $Dx = Ax + E(t)$ with $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}$ and $E(t) = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$.

9. (10 points): Determine whether the vector functions

$$h_1(t) = \begin{pmatrix} 3e^{4t} \\ e^{4t} \end{pmatrix} \quad \text{and} \quad h_2(t) = \begin{pmatrix} 3e^{-4t} \\ e^{-4t} \end{pmatrix}$$

generate the general solution of the system

$$x_1' = 5x_1 - 3x_2,$$
$$x_2' = 3x_1 - 5x_2.$$

Justify your answer.

Solution: (a) We can check that $h_2(t)$ does not satisfy the system,

(b) The Wronskian of these two vector functions is 0 for all t.

Either of these imply that the given vector functions **do not generate** the general solution.