Mathematics 51 **Differential Equations**
Examination I **Spring 2013**

Tufts UNIVERSITY

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. *Please box your answers and cross out any work you do not want graded.* Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (15 points) Consider the differential equation $x\dfrac{dx}{dt} = -t^2$.

a. Is this differential equation linear? Explain!

Solution: No, because of the xx'-term—the differential equation can not be brought into the form $a_1(t)x' + a_0(t)x = E(t)$.

b. Find the solution for which $x(0) = -5$.

Solution: Separation of variables: $x^2/2 = \int x\,dx = -\int t^2\,dt = -t^3/3 + C$ or with different C $x^2 = -2t^3/3 + C$. Insert $t = 0$, $x = -5$ to get $25 = (-5)^2 = C$, so $x = -\sqrt{25 - 2t^3/3}$.

2. (5 points) Show that the functions t^3 and t^4 are solutions of $t^2x'' - 6tx' + 12x = 0$.

Solution: Plug them in:
$t^2 \cdot 3 \cdot 2t - 6t \cdot 3t^2 + 12t^3 = 0$ ✓ and $t^2 \cdot 4 \cdot 3t^2 - 6t \cdot 4t^3 + 12t^4 = 0$ ✓.

3. (5 points, no partial credit) Find all solutions of $(tD^2 - D)x = 0$ that are of the form t^α.

Solution: Plug in t^α to get $(\alpha(\alpha-1) - \alpha)t^{\alpha-1} = 0$ for all t, hence $\alpha = 0$ and $\alpha = 2$ are the only solutions, so 1 and t^2 are (all) solutions of the differential equation of the form t^α.

4. (5 points, no partial credit) Evaluate

$$\det\begin{pmatrix} 0 & 1 & 6 & 9 & 1 \\ 0 & 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 0 & 4 \\ 5 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

5. (10 points) A savings account pays 3% interest per year, compounded continuously. In addition, the income from another investment is credited to the account continuously, at the rate of $700 per year. Set up a differential equation to model this account.

Solution: $x' = 0.03x + 700$.

6. (5 points, no partial credit) Find the general solution of $(D-1)^2(D+1)x = 0$.

Solution: $c_1e^t + c_2te^t + c_3e^{-t}$. *[This should take 15 seconds.]*

7. (7 points, no partial credit) Find the general solution of $3(D^2 + D + 2)^2x = 0$.

Solution: $3e^{-t/2}((c_1 + c_2t)\cos\dfrac{\sqrt{7}t}{2} + (c_3 + c_4t)\sin\dfrac{\sqrt{7}t}{2})$. *[Should take 1 minute.]*

8. (8 points, no partial credit) Make a *simplified* guess for a particular solution of

$$(D-1)(D^2+1)^3(D+2)x = t^2e^t + e^{-t}\sin 3t + t.$$

Do not evaluate the constants.

Solution: $c_1te^t + c_2t^2e^t + c_3t^3e^t + c_4e^{-t}\sin 3t + c_5e^{-t}\cos 3t + c_6 + c_7t.$ *[Should take 1 min.]*

9. (15 points) Find (and simplify where possible) the general solution of $x'' - 2x' + x = e^t/t^2$. (*Check all your intermediate answers carefully; no credit for work based on wrong prior steps.*)

Solution: General solution of the associated homogeneous differential equation $x'' - 2x' + x = 0$: $H(t) = c_1e^t + c_2te^t$; variation of parameters gives the solution $p(t) = c_1(t)e^t + c_2(t)te^t$ with

$$\begin{aligned} c_1'(t)e^t + c_2'(t)\ t\qquad e^t &= 0 \\ c_1'(t)e^t + c_2'(t)(t+1)e^t &= e^t/t^2, \end{aligned} \quad \text{or} \quad \begin{aligned} c_1'(t) + c_2'(t)\ t\qquad &= 0 \\ c_1'(t) + c_2'(t)(t+1) &= 1/t^2, \end{aligned} \quad \text{or} \quad \begin{aligned} c_1'(t) + c_2'(t)t &= 0 \\ c_2'(t)\ &= 1/t^2, \end{aligned}$$

so $c_2'(t) = 1/t^2$, $c_1'(t) = -1/t$, hence $c_1(t) = -1/t$, $c_2(t) = -\ln t$, and $p(t) = -e^t\ln t - \dfrac{1}{t}te^t$. Therefore $x(t) = c_1e^t + c_2te^t - e^t\ln t$.

10. (10 points)

a. Compute the Wronskian of $h_1(t) = te^t$ and $h_2(t) = t^2e^t$ at $t = 1$.

Solution: $W[te^t, t^2e^t] = \begin{vmatrix} te^t & t^2e^t \\ (t+1)e^t & (t^2+2t)e^t \end{vmatrix} = \begin{vmatrix} te^t & t^2e^t \\ e^t & 2te^t \end{vmatrix} = e^2$ for $t = 1$.

b. Are these 2 functions linearly independent?

Solution: Yes, because $e^2 \neq 0$.

11. (10 points) Are the functions t^5, $|t|^5$ linearly independent on $(-\infty, \infty)$? *Justify your conclusion.*

Solution: They are: If $c_1t^5 + c_2|t|^5 = 0$ for all t then for $t = 1$ we get $c_1 + c_2 = 0$ and for $t = -1$ we get $-c_1 + c_2 = 0$. Adding and subtracting these two equations gives $c_1 = c_2 = 0$.

12. (5 points) Suppose $f(t)$ is continuous. Solve the initial-value problem $x' + f(t)x = 0, \quad x(1) = 0$. (Hint: Think before applying standard techniques.)

Solution: By inspection, $x(t) = 0$ (for all t) is a solution that satisfies the initial condition.

One could get this by separation of variables (but why would you?): Rewrite as $\dfrac{dx}{dt} = -f(t)x$ and separate variables to get $\dfrac{dx}{x} = -f(t)dt$ (if $x \neq 0$). Integrating (and omitting absolute values), we get $\ln x = -\int f(t)\,dt + C$, so $x = e^{C-\int f(t)\,dt} = Ae^{-\int f(t)\,dt}$ for some constant A (which used to be positive but in retrospect does not have to be). To satisfy the initial condition, we must take $A = 0$, which gives $x(t) = 0$ for all t.

END OF EXAMINATION