Math 38 | Tufts University | Monday, February 13, 2012
Differential Equations | Department of Mathematics | 12:00–1:20pm
Exam I

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. A correct answer with no work may not necessarily score any points. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature, you are pledging that you have neither given nor received assistance on the exam. Any violations will be reported to the appropriate dean, and will result in an F for the course.
Please start each problem on a new page.

1. (10 points) Compute

$$det \begin{bmatrix} 0 & -1 & 3 & 2 \\ 5 & 0 & 2 & 1 \\ 6 & 0 & 0 & 1 \\ 2 & 0 & 0 & 3 \end{bmatrix}$$

2. (20 points) Consider the first order o.d.e

$$(A) \qquad (t-1)\frac{dx}{dt} = -x$$

   (a) Find the largest rectangular region in the tx-plane that contains $(0,0)$ on which the hypotheses of the existence and uniqueness theorem (E & UT) hold.
   (b) Do the hypotheses of the E & UT hold for the point $(1,0)$?
   (c) Do you have a solution of (A) with $x(1) = 0$?
   (d) Is it unique? Justify.

3. (10 points) Show that $h_1(t) = e^t$ and $h_2(t) = e^{2t}$ are linearly independent.

4. (14 points) Show that the functions $|t|$, t and t^2 for $-\infty < t < \infty$ are linearly independent.

5. (28 points)
   (a) Check that $h_1(t) = \frac{1}{t}$ and $h_2(t) = \frac{2}{t^2}$ are solutions of

$$(H) \qquad (t^2D^2 + 4tD + 2)x = 0 \qquad t > 0$$

   (b) Find the general solution. Explain <u>why</u> this is the general solution.
   (c) Find a constant solution for

$$(N) \qquad (t^2D^2 + 4tD + 2)x = 1$$

   (d) Find a solution for (N) that satisfies $x(1) = 0$, $x'(1) = 0$. Is it unique? Justify.

6. (18 points) Solve the system

$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 - x_5 &= 2 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 - 2x_5 &= 1 \\ -x_1 \qquad - x_3 + 2x_4 + x_5 &= 1 \end{aligned}$$

Solutions to Exam 1, Math 38

1. $\det\begin{bmatrix} 0 & -1 & 3 & 2 \\ 5 & 0 & 2 & 1 \\ 6 & 0 & 0 & 1 \\ 2 & 0 & 0 & 3 \end{bmatrix} = -32$

2 (A) $(t-1)\dfrac{dx}{dt} = -x$

$$x' = \frac{-x}{t-1} = f(t,x) \qquad \frac{\partial f}{\partial x} = \frac{-1}{t-1}$$

Both f and $\partial f/\partial x$ are discontinuous at $t=1$

a) $-\infty < t < 1$, $-\infty < x < \infty$

b) No because f and $\partial f/\partial x$ are discontinuous when $t=1$

c) $x(t)=0$ is a solution of (A) satisfying $x(1)=0$

d) Try separation of variables. If x is not the zero function

$$\ln|x| = \int \frac{dx}{x} = \int \frac{-1}{t-1}\,dt = \ln\left(\frac{1}{|t-1|}\right) + C$$

So $x = \dfrac{K}{|t-1|}$ If $K \neq 0$ you cannot plug in $t=1$

so the solution in c) is unique.

3) You can use the Wronskian

$$\begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = e^{3t} \neq 0$$

so e^t and e^{2t} are independent

4) You may not use the Wronskian because $|t|$ has no derivative at 0. Try

$$x|t| + yt + zt^2 = 0 \quad \text{for all } t$$

So $t = 1$ gives

$$\left.\begin{aligned} x + y + z &= 0 \\ x - y + z &= 0 \\ 2x + 2y + 4z &= 0 \end{aligned}\right\} (*)$$

$t = -1$

$t = 2$

Now $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 4 \end{vmatrix} = -2 \neq 0$ so the only solution to the system $(*)$ is $x = y = z = 0$ and so $|t|, t$ and t^2 are independent

5) a) $(t^2D^2 + 4tD + 2)t^{-1} = t^2(2t^{-3}) + 4t(-t^{-2}) + 2t^{-1}$

$= 2t^{-1} - 4t^{-1} + 2t^{-1} = 0$

$(t^2D^2 + 4tD + 2)\,2t^{-2} = t^2(12t^{-4}) + 4t(-4t^{-3}) + 2(2t^{-2})$

$= 12t^{-2} - 16t^{-2} + 4t^{-2} = 0$

b) $h_1 = \frac{1}{t}$ and $h_2(t) = \frac{2}{t^2}$ are independent because

$W\left(\frac{1}{t}, \frac{2}{t^2}\right) = -\frac{2}{t^4} \neq 0$ on $(0, \infty)$. They are solutions by a)

Thus $c_1 \frac{1}{t} + c_2 \frac{2}{t^2}$ is the general solution of A

which is a second order equation and so it needs two independent

solutions to generate the general solution.

c) If x is constant $Dx = D^2 x = 0$ and
$(t^2 D^2 + 4tD + 2)x = t^2 \cdot 0 + 4t \cdot 0 + 2x = 1$ gives
$x(t) = 1/2$ is a constant solution of (N)

d) Consequently $x = \frac{1}{2} + c_1 \frac{1}{t} + c_2 \frac{2}{t^2}$ is the general

solution and $x(1) = 0$ gives $\frac{1}{2} + c_1 + 2c_2 = 0$

We have $x'(t) = -\frac{c_1}{t^2} - \frac{4c_2}{t^3}$ and $x'(1) = 0$ gives

$-c_1 - 4c_2 = 0$ So we have the system with matrix

$\left(\begin{array}{cc|c} 1 & 2 & -1/2 \\ -1 & -4 & 0 \end{array}\right)$ which reduces to $\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1/4 \end{array}\right)$

So $c_1 = -1$ and $c_2 = 1/4$ give the unique solution to N
with $x(0) = x'(0) = 0$.

6) $\left(\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & -1 & 2 \\ 2 & 2 & 2 & -3 & -2 & 1 \\ -1 & 0 & -1 & 2 & 1 & 1 \end{array}\right) \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & -1 & 2 \\ 0 & -2 & 0 & -1 & 0 & -3 \\ 0 & 2 & 0 & 1 & 0 & 3 \end{array}\right)$

rows 2 and 3 are proportional. We may eliminate R_2 and
change it by a row of zeros at the bottom and put R_3 in the
second row

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & -1 & 2 \\ 0 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -2 & -1 & -1 \\ 0 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccccc|c} \boxed{1} & 0 & 1 & -2 & -1 & -1 \\ 0 & \boxed{1} & 0 & 1/2 & 0 & 3/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Pivots   Free variables

$$\begin{aligned} x_1 &= -x_3 + 2x_4 + x_5 - 1 \\ x_2 &= \quad - \tfrac{1}{2}x_4 \quad + 3/2 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{general homog sol'n}} + \underbrace{\begin{bmatrix} -1 \\ 3/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular}}$$

$x_3 = x_4 = x_5 = 1$ gives another particular: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$