Tufts University

Exam 1 | Department of Mathematics | February 14, 2011

Mathematics 38

1. (14 points) Consider the equation

$$tx' + (1-t)x^2 = 0$$

Please, answer the following questions (NO PARTIAL CREDIT):

(a) Is the given differential equation separable?

ANS: YES

(b) Is it linear?

ANS: NO

(c) Give the largest interval containing $t = 1$ where the equation is normal.

ANS: $(0, \infty)$

(d) Write the equation in the standard form.

$x^{'} = -\frac{1-t}{t}x^2$

(e) Find the general solution.

Separating variables and integrating, $-\frac{1}{x} = t - \ln|t| + C$. *ANS:* $x = \frac{1}{\ln|t|-t-C}$, $t \neq 0$, *and* $x = 0$.

(f) Find a solution satisfying the initial condition $x(1) = -2$.

ANS: $x = \frac{1}{\ln|t|-t+1/2}$

(g) Why is the solution satisfying the initial condition $x(1) = -2$ unique? (Please, refer to an appropriate theorem).

Due to the existence and uniqueness theorem.

2. (10 points) Consider the differential equation

$$(t-1)\frac{dx}{dt} = x.$$

(a) Determine the largest rectangular region of the (t, x)-plane that contains the point $(0, 2)$ and on which the hypotheses of the existence and uniqueness theorem hold for the given o.d.e.

ANS: $(-\infty, 1) \times (-\infty, \infty)$

(b) Determine whether there is no solution, a unique solution, or more than one solutions passing through the point $(1, 2)$.

If there is a solution $x(t)$ *then at* $t = 1$ *one has* $(1-1)x^{'}(1) = x(1)$, *or* $0 = 2$. *This contradiction shows that there is no solution passing through the point* $(1, 2)$. ANS: NO SOLUTION.

3 (10 points) Solve the following first order linear differential equation

$$2x^{'} - x = te^{t}.$$

(i) The general solution of the homogeneous equation is $x = Ce^{\frac{1}{2}t}$;
(ii) A particular solution to the nonhomogeneous equation is $(t-2)e^{t}$;
(iii) ANS: $x = Ce^{\frac{1}{2}t} + (t-2)e^{t}$;

4 (8 points) (NO PARTIAL CREDIT)
(a) Determine whether the system of linear algebraic equations

$$x - 2y + z = 0$$

$$3x + 2y + z = 0$$

$$y - z = 0$$

has a unique solution, infinitely many solutions, or no solution;
Cramer's test implies that the system has a unique solution
(b) If the system has a unique solution, then find this solution.
ANS: *(By inspection)* $(0, 0, 0)$.

5. (6 points) Use the Wronskian test for independence to show that functions e^{at} and e^{bt}, with $a \neq b$, are linearly independent.
The Wronskian is $W(t) = \det \begin{pmatrix} e^{at} & e^{bt} \\ ae^{at} & be^{bt} \end{pmatrix} = (b-a)e^{(a+b)t} \neq 0$ *if* $a \neq b$.

6. (10 points) Use the exponential shift formula to compute the following expressions:
(a) $(D^2 + D - 5)[te^{2t}]$
ANS: $(D^2+D-5)[te^{2t}] = e^{2t}((D+2)^2+(D+2)-5)[t] = e^{2t}(D^2+5D+1)[t] = e^{2t}(5+t)$.
(b) $(D-5)^4[e^{5t}\sin(t)]$
ANS: $(D-5)^4[e^{5t}\sin(t)] = e^{5t}D^4[\sin(t)] = e^{5t}\sin(t)$.

7. (15 points) Consider the following nonhomogeneous second order differential equation

$$x^{''} - 2x^{'} + x = t^2.$$

(a) Find the general solution of the corresponding homogeneous equation;
ANS: $x = (C_1 + tC_2)e^{t}$
(b) Find a particular solution of the form $x(t) = At^2+Bt+C$ to the nonhomogeneous equation;
ANS: $x_p = t^2 + 4t + 6$
(c) Find the solution of the given equation satisfying the conditions

$$x(0) = x^{'}(0) = 0.$$

ANS: $x = (2t-6)e^t + t^2 + 4t + 6$

8. (7 points) Find the annihilator of smallest possible order for the function (NO PARTIAL CREDIT)

$$t^2 + e^t \cos 2t - 1$$

ANS: $A(D) = D^3(D^2 - 2D + 5)$

9. (20 points) Solve the following initial-value problem

$$(D^2 - 1)(D^2 + 1)x = 0$$

$$x(0) = 1,\ x'(0) = x''(0) = x'''(0) = 0.$$

The general solution is $x = C_1 e^t + C_2 e^{-t} + C_3 \cos(t) + C_4 \sin(t)$.
Initial conditions give

$$\begin{aligned} C_1 + C_2 + C_3 &= 1, \\ C_1 - C_2 + C_4 &= 0, \\ C_1 + C_2 - C_3 &= 0, \\ C_1 - C_2 - C_4 &= 0. \end{aligned}$$

Adding first and third equations, and then second and forth equations, we have

$$\begin{aligned} C_1 + C_2 &= 1/2, \\ C_1 - C_2 &= 0. \end{aligned}$$

This implies $C_1 = C_2 = 1/4$, *and from second and third equations,* $C_3 = 1/2$ *and* $C_4 = 0$.

ANS: $x = \frac{1}{4}e^t + \frac{1}{4}e^{-t} + \frac{1}{2}\cos(t)$.