Mathematics 38 **Differential Equations**
Examination I **October 3, 2011**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

Please write the solutions to problems 1–5 on the inside (blue!) front cover of your exam book.

1. (3 points each, no partial credit) For each of the differential equations below determine the order, determine whether the differential equation is linear, and if so, whether it is homogeneous.

a. $t^4\dfrac{d^3x}{dt^3} + t\dfrac{dx}{dt} - x - t^7 = 0$

b. $x^8\dfrac{dx}{dt} + \dfrac{d^7x}{dt^7} = x + t^9$

c. $\left(\dfrac{dx}{dt}\right)^5 + \dfrac{d^4x}{dt^4} - t^3x^7 + t^7 = 0$

d. $(x')^2x''' = x^4x'' + t^5x'$

Solution: **a.** 3, l, n, **b.** 7, n, **c.** 4, n, **d.** 3, n.

2. (3 points each, no partial credit) Find all real values of α for which the given function is a solution of the given differential equation.

a. $x = \alpha,\qquad \dfrac{d^7x}{dt^7} + \dfrac{dx}{dt} - x = 7$

b. $x = t^\alpha$, $t > 0,\qquad 16t^2xx'' + 3x^2 = 0$

c. $x = e^{\alpha t},\qquad x'\sqrt{x} = 2e^{3t}$

Solution: **a.** -7, **b.** 1/4, 3/4, **c.** 2.

3. (1 point each) For each of the following differential equations state whether it is normal on $0 < t < 2$.

a. $(t-1)\dfrac{dx}{dt} - 5x = 3t$

b. $3\dfrac{dx}{dt} - 5x = \csc \pi t$

c. $t\dfrac{dx}{dt} + e^t x = \sin t$

d. $t\sin t\dfrac{dx}{dt} + \pi x = \ln t$

Solution: **a.** no, **b.** no, **c.** yes, **d.** yes.

4. (5 points) Evaluate the determinant $\det\begin{pmatrix} 0 & 1 & 6 & 9 & 1 \\ 0 & 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 0 & 4 \\ 5 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Solution: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

5. (5 points) What is the Wronskian of $h_1(t) = te^t$ and $h_2(t) = t^2e^t$ at $t = 0$?

Solution: $W[te^t, t^2e^t](0) = \begin{vmatrix} 0 & 0 \\ * & * \end{vmatrix} = 0$. (Generally, $W[te^t, t^2e^t] = \begin{vmatrix} te^t & t^2e^t \\ (t+1)e^t & (t^2+2t)e^t \end{vmatrix}$.)

6. (5 points) Solve the initial-value problem $x\dfrac{dx}{dt} = -t^2, \quad x(0) = -5.$

Solution: Separation of variables: $x^2/2 = \int x\,dx = -\int t^2\,dt = -t^3/3 + C$. Insert $t = 0$, $x = -5$ to get $25 = (-5)^2 = 2C$, so $x = -\sqrt{25 - 2t^3/3}$.

7. (10 points) At the point $(t_0, \alpha) = (1, 2)$ the Existence and Uniqueness Theorem does not apply to the differential equation

$$(t-1)\frac{dx}{dt} = -x.$$

Determine whether there are no solutions, more than one solution or a unique solution of the differential equation with the initial value $x(1) = 2$. Explain!

Solution: Plug in $t = 1$, $x = 2$ to get $0 = -2$, which is impossible—there is no solution.

8. (10 points) Consider the functions t^3 and $|t^3|$ defined on $-\infty < t < \infty$. Are they linearly independent? Explain!

(Hint: Computing the Wronskian may *not* be the best approach.)

Solution: If $c_1t^3 + c_2|t^3| = 0$ for all t, then we can insert $t = 1$ to get $c_1 + c_2 = 0$ and $t = -1$ to get $-c_1 + c_2 = 0$; together these give $c_1 = 0$ and $c_2 = 0$, so these functions are linearly independent.

9. (15 points) Consider the first-order differential equation

(*) $$\frac{dx}{dt} = x^2(x+2)(x-1)^3.$$

a. Find all equilibria and draw the phase portrait of (*).

Solution: The equilibria are $-2, 0, 1$; the phase portrait is $\rightarrow\!-\!\bullet\!-\!\leftarrow\!-\!\bullet\!-\!\leftarrow\!-\!\bullet\!\rightarrow\!-$.

b. Which equilibria are attractors, which are repellers, and which are neither? Which are stable and which are unstable?

Solution: -2 is an attractor, 1 is a repeller, 0 is neither; -2 is stable, 0 and 1 are unstable.

c. Does there exist a value x_0 such that the solution $x(t)$ of (*) with initial condition $x(0) = x_0$ approaches ∞ as $t \to \infty$? If your answer is "yes", please provide a specific value that works; if your answer is "no", explain why this is so.

Solution: Yes, 17. (Or any other number greater than 1.)

d. Does there exist a value x_0 such that the solution $x(t)$ of (*) with initial condition $x(0) = x_0$ satisfies $\lim_{t\to\infty} x(t) = -1$? If your answer is "yes", please provide a specific value that works; if your answer is "no", explain why this is so.

Solution: There is no such initial value because -1 is not an equilibrium.

10. (10 points) Find the general solution of $\dfrac{d^5x}{dt^5} - 2\dfrac{d^4x}{dt^4} + \dfrac{d^3x}{dt^3} = 0.$

Solution: $D^5 - 2D^4 + D^3 = D^3(D^2 - 2D + 1) = D^3(D-1)^2$, so

$$x(t) = c_1 + c_2t + c_3t^2 + c_4e^t + c_5te^t.$$

11. (15 points) Find the general solution of $x'' + x = 2\sec t, \qquad 0 < t < \pi/2.$

Solution: $h(t) = c_1\cos t + c_2\sin t$. Variation of parameters gives $\begin{cases} k_1'(t) = -2\tan t, \\ k_2'(t) = 2, \end{cases}$ hence

$$k_1(t) = 2\ln\cos t \text{ and } k_2(t) = 2t,$$

so $p(t) = 2\cos t\ln\cos t + 2t\sin t$ and

$$x(t) = c_1\cos t + c_2\sin t + 2\cos t\ln\cos t + 2t\sin t.$$

END OF EXAMINATION