Mathematics 38 **Differential Equations**
Final Exam **December 16, 2011**

Tufts UNIVERSITY

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (5 points each, no partial credit) In parts **a., b., c.**, find all real values of α, if any, for which the given function is a solution of the given differential equation.

a. $x = \alpha$, $\quad \dfrac{d^7x}{dt^7} + \dfrac{dx}{dt} - x = 7$

b. $x = t^\alpha$, $t > 0$, $\quad 16t^2xx'' + 3x^2 = 0$

c. $x = e^{\alpha t}$, $\quad x'\sqrt{x} = 2e^{3t}$

d. Use the method of undetermined coefficients (the "annihilator method") to determine a simplified guess for a particular solution of

$$(D+3)^2(D^2+1)^4x = \sin t.$$

Obtain the simplest form possible, and leave your answer in terms of the undetermined coefficients.

Do not try to determine the coefficients!

e. Consider the functions t^3 and $|t^3|$ defined on $-\infty < t < \infty$. Are they linearly independent? Give a succinct and definitive reason.

f. The vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$, $\vec{v}_5$ are linearly independent. Decide whether $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ are linearly independent. Give a succinct and definitive reason.

g. Suppose $g(t)$ is a continuous function. Find a solution of the initial-value problem

$$x' = g(t)x, \quad x(1) = 0.$$

Simplify it as much as possible. (Hint: Think before applying standard techniques.)

2. (5 points, no partial credit) Choose one answer.

$$\det\begin{pmatrix} 5 & 1 & \sin t & t^2+3 & 1 \\ 0 & 4 & e^t & e^t & 0 \\ 0 & 0 & 3 & \ln t & 8 \\ 0 & 0 & 0 & 2 & \sqrt{t} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

a. 5, **b.** $2\sqrt{t}$, **c.** 120, **d.** $120 - \sin t - \frac{1}{4}e^t - \frac{1}{4}(t^2+3)e^t$.
e. None of the above.

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3. (10 points) Find the general solution of $\cos t \, \dfrac{dx}{dt} + x \sin t = \cos t \sin t, \quad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}.$

4. (15 points) Use the Laplace transform to solve

$$(D-2)(D-1)^2 x = -2e^t \qquad \text{with} \quad x(0) = x'(0) = 0 \text{ and } x''(0) = 2.$$

5. (15 points) Find the general solution of $D\vec{x} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \vec{x}.$

6. (20 pts) Consider the system of differential equations

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x.$$

a. Check whether $x^2 + y^2$ is a Lyapunov function,

b. check whether $x^2 + y^2$ is a constant of motion,

c. find the equilibria. (Make absolutely sure to get them right; otherwise there will be very little partial credit.)

For each equilibrium

d. determine its stability,

e. decide whether it is an attractor, a repeller, or neither.

f. Determine whether there are closed integral curves,

g. draw a plausible phase portrait.

END OF EXAMINATION