	Tufts University	
MATH 51	Department of Mathematics	Monday, Oct. 12, 2015
Differential Equations	Exam 1	12:00pm – 1:20pm

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. *Please box your answers and cross out any work you do not want graded.* Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. **There are two pages to this exam, totaling 6 questions.** *Good luck!*

1. Warm up:

   (a) (1 point each) Classify the following ODE's by order and whether it is linear or nonlinear. If the ODE is linear, also indicate if it is homogeneous or nonhomogeneous. Finally, if the ODE is linear, give the intervals where the ODE is normal.

$$\text{(i)}\quad \left(\frac{d^3x}{dt^3}\right)^3 + \frac{dx}{dt} = 0$$

$$\text{(ii)}\quad x\frac{dy}{dx} - y = \frac{\sqrt{1-e^{x/3}}}{8}$$

$$\text{(iii)}\quad \frac{t}{t-1}\frac{d^2x}{dt^2} = \sqrt{t}x$$

   intervals where the ODE is normal.

   Answer:

   (i) $\boxed{\text{third-order, nonlinear}}$

   (ii) $\boxed{\text{first-order, linear, nonhomogeneous, normal when } x < 0}$

   (iii) $\boxed{\text{second-oder, linear, homogeneous, normal for } t \in (0,1) \cup (1,\infty)}$

   (b) (2 points) True or False: The function $g(t) = \sqrt{2t}$ is a solution to the ODE

$$(t^{3/2}D^2 + \sqrt{t})x = \frac{4t-1}{\sqrt{8}}.$$

   Answer: $\boxed{\text{True}}$

   (c) (2 points) True or False: The annihilator of $t^2\cos(t) + t^2$ is $(D^2+1)^3 + D^3$.
   Answer: $\boxed{\text{False}}$ (annihilator is $(D^2+1)^3D^3$)

   (d) (2 points) True or False: Separation of variables can solve some nonlinear ODEs.
   Answer: $\boxed{\text{True}}$

EXAM CONTINUES ON REVERSE

2. Consider the ODE

$$(D^2 + 4)x = \cos(t). \tag{1}$$

(a) (15 points) What is the general solution to equation (1)?

Answer: The solution to the associated homogeneous equation is

$$x_H(t) = c_1 \cos(2t) + c_2 \sin(2t) \tag{2}$$

The forcing function $\cos(t)$ is annihilated by $D^2 + 1$. Applying the annihilator to (2) yields

$$(D^2 + 1)(D^2 + 4)x = 0,$$

so guess for particular solution is

$$x_P(t) = A\cos(t) + B\sin(t).$$

Substituting the guess for $x_P(t)$ into (2) yields the equation

$$3A\cos(t) + 3B\sin(t) = \cos(t),$$

so solutions is $A = \frac{1}{3}$ and $B = 0$. Finally, the general solution to (2) is

$$\begin{aligned} x(t) &= x_H(t) + x_P(t) \\ &= \boxed{c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{3}\cos(t)} \end{aligned} \tag{3}$$

(b) (6 points) What is the specific solution to equation (1) satisifying

$$x(0) = x'(0) = 0?$$

Answer: Substituting the general solution (3) into $x(0) = 0$ yields

$$c_1 + \frac{1}{3} = 0 \Rightarrow c_1 = -\frac{1}{3}.$$

Likewise, substituting $x'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{1}{3}\sin(t)$ into $x'(0) = 0$ yields

$$2c_2 = 0 \Rightarrow c_2 = 0.$$

Therefore, the specific solution satisfying the given initial conditions is

$$\boxed{x(t) = -\frac{1}{3}\cos(2t) + \frac{1}{3}\cos(t)}$$

3. (15 points) Solve the following differential equation using any method you see fit.

$$x' + \cos(t)x = \cos(t)$$

Answer: *This ODE is separable, but if you don't see that, it can be solved using variation of parameters.* The associated homogeneous equation is

$$x' + \cos(t)x = 0$$

Using separation of variables on this yields

$$\begin{aligned} \frac{dx}{x} &= -\cos(t)\,dt \\ \Rightarrow \ln|x| &= -\sin(t) + c \\ \Rightarrow x_H(t) &= c\,e^{-\sin(t)} \end{aligned}$$

Substituting the guess $x(t) = k(t)e^{-\sin(t)}$ yields

$$\begin{aligned} \left(k'(t)e^{-\sin(t)} - \cos(t)k(t)e^{-\sin(t)}\right) + \cos(t)k(t)e^{-\sin(t)} &= \cos(t) \\ \iff k'(t)e^{-\sin(t)} &= \cos(t) \\ \Rightarrow k'(t) &= \cos(t)e^{\sin(t)} \end{aligned}$$

Integrating yields

$$k(t) = e^{\sin(t)} + c.$$

So, the general solution is

$$\begin{aligned} x(t) &= (e^{\sin(t)} + c)e^{-\sin(t)} \\ &= \boxed{c\,e^{-\sin(t)} + 1} \end{aligned}$$

4. (15 points) Are the functions $h_1(t) = t^2 + e^{t-1}$, $h_2(t) = 2t^2$, and $h_3(t) = 4e^t$ linearly independent? Justify your answer.

Answer: We observe $h_1(t) = \frac{1}{2}h_2(t) + \frac{e^{-1}}{4}h_3(t)$, and so $c_1 1 = -1$, $c_2 = \frac{1}{2}$, and $c_3 = \frac{e^{-1}}{4}$ is a nonzero solution to $c_1h_1(t) + c_2h_2(t) + c_3h_3(t) = 0$. Therefore, the functions are linearly dependent.

Taking the Wronskian hurts, but if you didn't see the linear dependence right away, you should get zero. For example,

$$\begin{aligned}
\left|\begin{bmatrix} t^2+e^{t-1} & 2t^2 & 4e^t \\ 2t+e^{t-1} & 4t & 4e^t \\ 2+e^{t-1} & 4 & 4e^t \end{bmatrix}\right| &= 4e^t\left|\begin{bmatrix} 2t+e^{t-1} & 4t \\ 2+e^{t-1} & 4 \end{bmatrix}\right| - 4e^t\left|\begin{bmatrix} t^2+e^{t-1} & 2t^2 \\ 2+e^{t-1} & 4 \end{bmatrix}\right| \\
&\quad + 4e^t\left|\begin{bmatrix} t^2+e^{t-1} & 2t^2 \\ 2t+e^{t-1} & 4t \end{bmatrix}\right| \\
&= 4e^t\big[\big(8t + 4e^{t-1} - 8t - 4te^{t-1}\big)\ldots \\
&\quad - \big(4t^2 + 4e^{t-1} - 4t^2 - 2t^2e^{t-1}\big)\ldots \\
&\quad + \big(4t^3 + 4te^{t-1} - 4t^3 - 2t^2e^{t-1}\big)\big] \\
&= 0
\end{aligned}$$

(Colors are used to show which terms cancel.)

5. (20 points) Is the collection of solutions

$$x(t) = c_1e^t + c_2te^t + c_3(t-1)e^t + e^{-t}$$

the general solution to

$$(D^3 - 2D^2 + D)x = -4e^{-t}?$$

If not, give an initial condition of the form

$$\begin{aligned} x(t_0) &= \alpha_0 \\ x'(t_0) &= \alpha_1 \\ x''(t_0) &= \alpha_2 \end{aligned}$$

which cannot be satisfied by the proposed collection of solutions.

Answer: Let $x_p = e^{-t}$, $h_1(t) = e^t$, $h_2(t) = t\,e^t$, and $h_3(t) = (t-1)e^t$. Also, let $L = D(D-1)^2$. The characteristic polynomial of L has a double root equal to 1, so we know $h_1(t)$ and $h_2(t)$ are solutions to the associated homogeneous ODE $L[x] = 0$. Since $h_3(t) = h_2(t) - h_1(t)$ and L is linear, we also know that $h_3(t)$ is a solution to $L[x] = 0$. (You could also show these functions were solutions to $L[x] = 0$ by computing $L[h_j]$ and showing it equals zero for $j = 1, 2, 3$.) Likewise, we can see that

$$L[x_p] = -e^{-t} - 2e^{-t} - 1e^{-t} = -4e^{-t},$$

so x_p is a particular solution to the nonhomogeneous ODE $L[x] = -4e^{-t}$. It follows that the proposed collection of solutions would generate the general solution if $h_1(t)$,

$h_2(t)$, and $h_3(t)$ were linear independent. However, since $h_3(t) = h_2(t) - h_1(t)$, the functions are linearly dependent. Therefore, the proposed collection of solutions $x(t)$ is not the general solution.

To demonstrate there is an initial condition that is not satisfied, first compute the required derivatives of $x(t)$:

$$\begin{aligned} x(t) &= c_1 e^t + c_2 t e^t + c_3 (t-1) e^t + e^{-t} \\ x'(t) &= c_1 e^t + c_2 (t+1) e^t + c_3 t e^t - e^{-t} \\ x''(t) &= c_1 e^t + c_2 (t+2) e^t + c_3 (t+1) e^t + e^{-t} \end{aligned}$$

Then, let's choose $t_0 = 0$ for simplicity, yielding

$$\begin{aligned} x(0) &= c_1 - c_3 + 1 \\ x'(0) &= c_1 + c_2 - 1 \\ x''(0) &= c_1 + 2c_2 + c_3 + 1 \end{aligned}$$

It follows that the initial conditions are equivalent to

$$\begin{aligned} \tilde{\alpha}_0 &= c_1 - c_3 && (4) \\ \tilde{\alpha}_1 &= c_1 + c_2 && (5) \\ \tilde{\alpha}_2 &= c_1 + 2c_2 + c_3 && (6) \end{aligned}$$

where $\tilde{\alpha}_0 = \alpha_0 - 1$, $\tilde{\alpha}_1 = \alpha_1 + 1$, and $\tilde{\alpha}_2 = \alpha_2 - 1$. Subtracting (4) from (5) and subtracting (4) from (6) yields

$$\begin{aligned} \tilde{\alpha}_1 - \tilde{\alpha}_0 &= c_2 + c_3 && (7) \\ \tilde{\alpha}_2 - \tilde{\alpha}_0 &= 2c_2 + 2c_3 && (8) \end{aligned}$$

Combining these two equations by subtracting two-times (7) from (8) yields

$$\begin{aligned} & \tilde{\alpha}_2 - 2\tilde{\alpha}_1 + \tilde{\alpha}_0 = 0 \\ \iff & \alpha_2 - 2\alpha_1 + \alpha_0 = 4 \end{aligned}$$

It follows that if we choose α_0, α_1, and α_2 such that $\alpha_2 - 2\alpha_1 + \alpha_0 \neq 4$, then $x(t)$ cannot satisfy it. Such a set of initial conditions is

$$\begin{aligned} x(t_0) &= 0 \\ x'(t_0) &= 0 \\ x''(t_0) &= 0 \end{aligned}$$

(*Note that writing this set of initial conditions without any justification does not ear partial credit.)*

6. *(20) Given that* $h_1(t) = t^{-1}$ *and* $h_2(t) = t^2$ *generate the complete solution to the ODE*

$$(tD^2 - 2/t)x = 0, \quad t > 0,$$

what is the general solution to

$$(tD^2 - 2/t)x = t, \quad t > 0?$$

Answer: *Writing the ODE in standard form gives* $q(t) = 1$. *Note, t*

$$W[h_1, h_2](t) = \left| \begin{bmatrix} t^{-1} & t^2 \\ -t^{-2} & 2t \end{bmatrix} \right| = 2 + 1 = 3$$

Then

$$\begin{aligned} k_1'(t) &= \frac{-q(t)h_2(t)}{W(t)} = -\frac{t^2}{3} \Rightarrow k_1(t) = -\frac{t^3}{9} + c_1 \\ k_2'(t) &= \frac{q(t)h_1(t)}{W(t)} = -\frac{t^{-1}}{3} \Rightarrow k_2(t) = \frac{1}{3}\ln(t) + c_2. \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} x(t) &= k_1(t)h_1(t) + k_2(t)h_2(t) \\ &= \boxed{c_1 t^{-1} + c_2 t^2 - \frac{t^2}{9} + \frac{1}{3}t^2\ln(t)} \end{aligned}$$