Math 51 | Tufts University | Monday, May 5, 2014
Differential Equations | Department of Mathematics | 8:30–10:30am
Final Exam

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. A correct answer with no work may not necessarily score any points. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature, you are pledging that you have neither given nor received assistance on the exam. Any violations will be reported to the appropriate dean, and will result in an F for the course.

1. (16 points) The general solution to

$$(E) \qquad (D^2 - 1)x = 1$$

is $x(t) = -1 + c_1 e^t + c_2 e^{-t}$. You do not have to verify this.

   (a) Write a first-order system (S) that is equivalent to (E).

   (b) By any method, find the general solution **to** $(\mathbf{S})$.

2. (9 points) Find the solution(s) to

$$(t^2 - 1)\frac{dx}{dt} + x = 0, \qquad t > 1, \qquad x(2) = 1$$

3. (15 points) Find the solution to

$$x' = x + t^2 \qquad x(0) = 1$$

by the method of Laplace transforms. No credit by any other method or if you leave indicated convolutions in your answer.

4. (15 points) Find the general solution of

$$D\vec{x} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x}$$

5. (15 points) Consider the equation

$$\frac{dx}{dt} = x^2(1 - x^2)$$

   (a) Find the equilibrium points and identify them as attractors, repellers, or neither.

   (b) Draw the phase portrait of the equation.

6. (10 points) Find the general solution of

$$D(D + 1)(D - 1)x = te^t$$

7. (20 points) Consider the system

$$\frac{dx}{dt} = 2xy - x^2 - x$$
$$\frac{dy}{dt} = 2xy - y^2 + y$$

and the function $E = x^2y - xy^2 + xy$.

(a) Is E a constant of motion? Give reasons.

(b) Is E a Lyanpunov function? Give reasons.

(c) Find all the equilibria of the system.

(d) Find the linearization at each equilibrium and the eigenvalues of that linearization. Decide what kind of phase portrait you have (if the Hartman-Grobman theorem works).

(e) Classify each equilibrium as stable or unstable.

(f) Classify each equilibrium as an attractor, a repeller, or neither.