1) $(D^2 - 4)x = 2(1-u_2)$ $\qquad x(0)=1, \quad x'(0)=0$

b) $\mathcal{L}\, 2(1-u_2) = 2\left(\frac{1}{s} \quad \frac{e^{-2s}}{s}\right)$

c)
$$\left.\begin{aligned} \frac{2}{s(s+2)(s-2)} &= \frac{-1/2}{s} + \frac{1/4}{s+2} + \frac{1/4}{s-2} \\ \frac{s}{(s+2)(s-2)} &= \frac{1/2}{s+2} + \frac{1/2}{s-2} \end{aligned}\right] \text{ Need}$$

$$\mathcal{L} D^2 x = s^2 \mathcal{L}x - s$$
$$\mathcal{L}\, 4x = 4\mathcal{L}x$$
$$\mathcal{L}(D^2-4)x = (s^2-4)\mathcal{L}x - s$$

$$(s+2)(s-2)\mathcal{L}x = s + 2\left(\frac{1}{s} - \frac{e^{-2s}}{s}\right)$$

$$\mathcal{L}x = \frac{s}{(s+2)(s-2)} + \frac{2}{s(s+2)(s-2)}\left(1-e^{-2s}\right)$$

$$= \frac{1/2}{s+2} + \frac{1/2}{s-2} + \left(\frac{-1/2}{s} + \frac{1/4}{s+2} + \frac{1/4}{s-2}\right)\left(1-e^{-2s}\right)$$

$$x = 1/2\, e^{-2t} + 1/2\, e^{2t} - \frac{1}{2} + 1/4\, e^{-2t} + 1/4\, e^{+2t} +$$

$$\cdot\, u_2(t)\left[\frac{1}{2} - \frac{1}{4} e^{-2(t-2)} - \frac{1}{4} e^{2(t-2)}\right]$$

$$= -\frac{1}{2} + \frac{3}{4} e^{-2t} + \frac{3}{4} e^{2t} + u_2(t)\left[\frac{1}{2} - \frac{1}{4} e^{-2(t-2)} - \frac{1}{4} e^{2(t-2)}\right].$$

2) $\mathcal{L}^{-1} \frac{s}{s^2+4s+5} = \mathcal{L}^{-1} \frac{s}{(s+2)^2+1} = \mathcal{L}^{-1} \frac{s+2}{(s+2)^2+1} - 2\mathcal{L} \frac{1}{(s+2)^2+1}$

$= e^{-2t}(\cos t - 2\sin t)$

3) $\begin{pmatrix} 1 & 2 & -1 \\ 5 & 3 & 9 \\ -1 & 4 & -11 \\ 0 & 2 & -4 \\ 2 & 1 & 4 \end{pmatrix}$ reduces to $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

b) Only 2 pivots so dependent

4) e-values (by inspection) 1, 4, 6

$A - I$ by inspection yields $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$A - 4I$ $\begin{pmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{pmatrix}$ reduces to $\begin{pmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ yields $\vec{v} = \begin{pmatrix} 2/3 \\ 1 \\ 0 \end{pmatrix}$

$A - 6I$ $\begin{pmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{pmatrix}$ reduces to $\begin{pmatrix} 1 & 0 & -8/5 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{pmatrix}$ yields $\vec{w} = \begin{pmatrix} 8/5 \\ 5/2 \\ 1 \end{pmatrix}$

G. Solution $c_1 e^{t} \vec{u} + c_2 e^{4t} \vec{v} + c_3 e^{6t} \vec{w}$

5) $\lambda = 1$ double $A - I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

generalized e-vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Solutions

$$e^t\left(\begin{pmatrix}1\\0\end{pmatrix}+t\begin{pmatrix}-1\\-1\end{pmatrix}\right)=e^t\begin{pmatrix}1-t\\-t\end{pmatrix}$$

$$e^t\left(\begin{pmatrix}0\\1\end{pmatrix}+t\begin{pmatrix}1\\1\end{pmatrix}\right)=e^t\begin{pmatrix}t\\1+t\end{pmatrix}$$

6) $A+I=\begin{pmatrix}0&-1&0\\1&0&1\\0&1&0\end{pmatrix}$ $\qquad (A+I)^2=\begin{pmatrix}-1&0&-1\\0&0&0\\1&0&1\end{pmatrix}$

and $(A+I)^3=0$ so generalized e-vectors $\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}$

Solutions

$$e^{-t}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}+t\begin{pmatrix}0\\1\\0\end{pmatrix}+\frac{t^2}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)$$

$$e^{-t}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}+t\begin{pmatrix}-1\\0\\1\end{pmatrix}+\frac{t^2}{2}\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$e^{-t}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}+t\begin{pmatrix}0\\1\\0\end{pmatrix}+\frac{t^2}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)$$

7) Homogeneous: e-values $-1,\ 3$

e-vector for -1: $\begin{pmatrix}-1\\2\end{pmatrix}$ or $\begin{pmatrix}1\\-2\end{pmatrix}$ e-vector for $+3$: $\begin{pmatrix}1\\2\end{pmatrix}$

Wronskian $\begin{pmatrix}+e^{-t} & e^{3t}\\-2e^{-t} & 2e^{3t}\end{pmatrix}$ $\qquad |W|=e^{2t}\begin{vmatrix}+1&1\\-2&2\end{vmatrix}=+4e^{2t}$

$$c'=\frac{1}{2}-\frac{1}{4}e^{-t}=\frac{\begin{vmatrix}e^{-t}&e^{3t}\\1&2e^{3t}\end{vmatrix}}{4e^{2t}}$$

$$c_2' = \frac{\begin{vmatrix} e^{-t} & e^{-t} \\ -2e^{-t} & 1 \end{vmatrix}}{4e^{2t}} = \frac{1}{4}e^{-3t} + \frac{1}{2}e^{-4t}$$

$$c_1 = \frac{t}{2} + \frac{1}{4}e^{-t} \qquad c_2 = -\frac{1}{12}e^{-3t} - \frac{1}{8}e^{-4t}$$

Put together

$$\left(\frac{t}{2} + \frac{1}{4}e^{-t}\right)e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \left(\frac{1}{12}e^{-3t} + \frac{1}{8}e^{-4t}\right)e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\frac{\left(2te^{-t} + e^{-2t}\right)}{4}\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \left(\frac{1}{12} + \frac{1}{8}e^{-t}\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$