Math 42 | Tufts University | Friday, May 13th, 2016
Calculus III | Department of Mathematics | 8:30 am–10:30 am

Final Exam Solutions

1 True

F is conservative, and thus the line integral around any closed curve is 0.

2 No

In order to conclude that, we would need this to hold for *all* smooth curves with the same starting and ending points, not just two.

3 False

$\vec{r}'(t) =< 1, 2t, 3t^2 >$, so $|\vec{r}'(t)| = \sqrt{1+4t^2+9t^4} \neq 1$.

4 No

$\nabla \cdot \vec{F}$ is a scalar, and you cannot dot a vector with a scalar.

5 (a) First we make an explicit function $F(x,y,z) = f(x,y) - z = 0$

$\nabla F =< -\frac{1}{1+x+y}, -\frac{1}{1+x+y}, -1 >$. Plugging in $(0,0)$ yields $< -1, -1, -1 >$.

The plane is then $-1(x-0) + -1(y-0) - 1(z-0) = -x + -y - z = 0$. Equivalently, $x+y+z = 0$

(b) Approach One: Use differentials

$f(0,1,0.2) = f(0,0) + \Delta z = 0 + \Delta z$.

$\Delta z \approx dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (-1)(0.1) + (-1)(0.2) = 0.3$

$f(0,1,0.2) \approx 0.3$

Approach Two: Use the linear approximation from part (a).

Near $(0,0)$, $f(x,y) \approx L(x,y) = -x - y$.

$f(0.1,0.2) \approx L(0.1,0.2) = -(0.1) - (0.2) = -0.3$

6 We seek to integrate $\iint_R 1 = \frac{1}{2}\oint_C (-y)dx + xdy$, where R is the region bounded by our curves and C is the boundary of the region, positively oriented.

The two curves intersect when $y^2 = 18 - y^2$, which means $y^2 = 9$ and $x = \pm 3$. We can parametrize our boundary C by the following:

$\vec{r_1}(t) =< 18 - t^2, t >, -3 \leq t < 3$, where $\vec{r_1}'(t) =< -2t, 1 >$

$\vec{r_2}(t) =< t^2, -t >, -3 \leq t < 3$, where $\vec{r_2}'(t) =< 2t, -1 >$

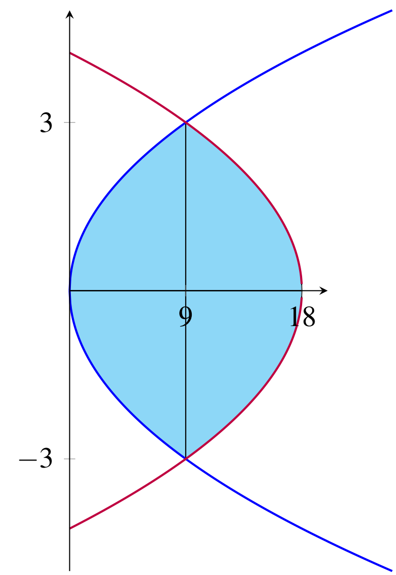
We then have that

$$\iint_R 1 = \frac{1}{2}\oint_C (-y)dx + xdy = \frac{1}{2}\oint_C < -y, x > \cdot T ds = \frac{1}{2}\int_{C_1} < -y, x > \cdot T ds + \frac{1}{2}\int_{C_2} < -y, x > \cdot T ds$$

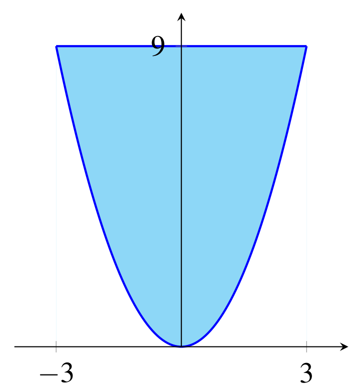
$$\frac{1}{2}\int_{C_1} < -y, x > \cdot T ds = \frac{1}{2}\int_{-3}^{3} < -t, 18 - t^2 > \cdot < -2t, 1 > dt = \frac{1}{2}\int_{-3}^{3} 2t^2 + 18 - t^2 dt = 63$$

$$\frac{1}{2}\int_{C_1} <-y,x> \cdot T\,ds = \frac{1}{2}\int_{-3}^{3} <t,t^2> \cdot <2t,-1>dt = \frac{1}{2}\int_{-3}^{3} 2t^2 - t^2 dt = 9$$

Therefore the area of the enclosed region is $63+9=72$.



7 The region we are integrating over is the following:



First we look for points inside the region:

$\nabla f = < y-3x^2, x+1 > = < 0,0 > \implies x=-1$ and $y=3$. Now we need to find the potential maxes/mins on our boundary, which can be parametrized by the following two functions:

$\vec{r}_1(t) = < t,t^2 >, -3 \leq t < 3.$

$\vec{r}_2(t) = < t,9 >, -3 < t \leq 3.$

$f(\vec{r}_1(t)) = t^3 + t^2 - t^3 = t^2$, which has a critical point at $t=0$.

$f(\vec{r}_2(t)) = 9t + 9 - t^3$, which has derivaive $9-3t^2$ and critical points $t = \pm\sqrt{3}$.

We now compare our critical points values and our endpoints

(x,y)	$f(x,y)$	max/min
$(-1,3)$	1	
$(0,0)$	0	
$(\sqrt{3},9)$	$9+6\sqrt{3}$	max
$(-\sqrt{3},9)$	$9-6\sqrt{3}$	min
$(3,9)$	9	
$(-3,9)$	9	

8 (a) to do this, we find the curl of $\vec{F}$ is 0 by the following:

$$\nabla \times \vec{F} = \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz+z & xy+y \end{bmatrix} =< (x+1)-(x+1), -(y-y), (z-z) >=< 0,0,0 >$$

(b) $\phi_x = yz$, so $\phi = \int \phi_x dx = xyz + C_1(y,z)$.

$\phi_y = xz + z = xz + C_1'(y,z)$, so $C_1(y,z) = \int z dy = yz + C_2(z)$.

$\phi_z = xy + y + C_2'(z) = xy + y$, so $C_2(z) = \int 0 dz = C_3$.

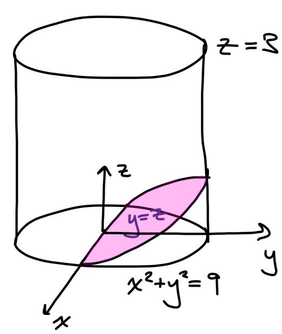
Thus $\phi = xyz + yz + C_3$.

(c)

$$\int_C \vec{F} d\vec{r} = \int_C \nabla \phi d\vec{r} = \phi(2,2,2) - \phi(3,3,3) = 12 - 36 = -24$$

(d) The fundamental theorem of line integrals tells us that the integral of a conservative vector field is the difference of the potential function at the endpoints.

9



Cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$$\int_0^{\pi} \int_0^3 \int_{r\sin\theta}^3 r dz dr d\theta + \int_{\pi}^{2\pi} \int_0^3 \int_0^3 r dz dr d\theta$$

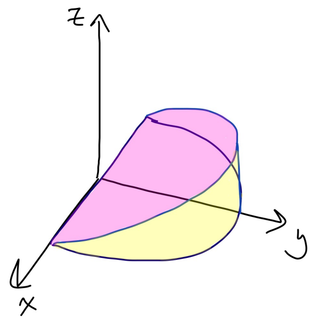
$$\int_{\pi}^{2\pi} \int_0^3 \int_0^3 r dz dr d\theta = \frac{27\pi}{2}$$

$$\int_0^{\pi} \int_0^3 \int_{r\sin\theta}^3 r dz dr d\theta = \int_0^{\pi} \int_0^3 3r - r^2 \sin\theta dr d\theta$$

$$\int_0^\pi \frac{27}{2} - 9\sin\theta d\theta = \frac{27}{2}\pi + 9\cos(\pi) - 0 - 9\cos(0) = \frac{27}{2}\pi - 18$$

So the volume is $27\pi - 18$.

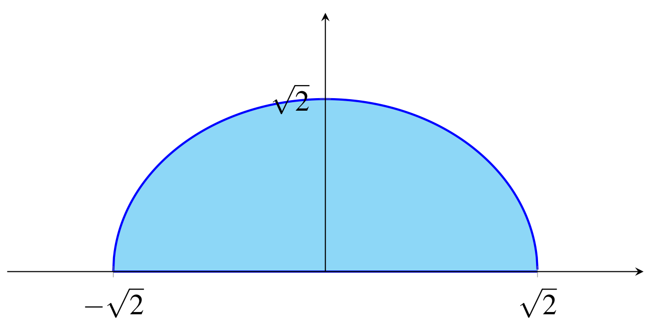
Alternatively, the shape is a cylinder with a small piece (shown below) cut out of it.



The cylinder ha volume $\pi r^2 h = 27\pi$ The small piece has volume

$$\int_0^\pi \int_0^3 \int_0^{r\sin\theta} r dz dr d\theta = \int_0^\pi \int_0^3 r^2 \sin\theta dr d\theta = \int_0^\pi 9\sin\theta d\theta = 18.$$

10 The region of interest is the following:



We can apply the circulation form of Green's Theorem to have that

$$\int_C (-3y - e^{x^2})dx + (x - e^{y^2})dy = \iint_D \frac{\partial}{\partial x}(x - e^{y^2}) - \frac{\partial}{\partial y}(-3y - e^{x^2})dA = \iint_D 1 + 3dA$$

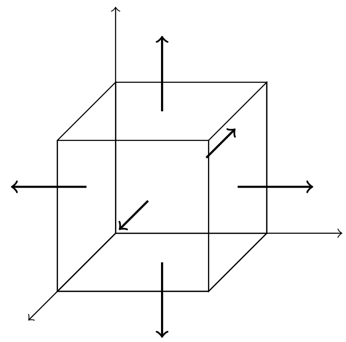
Then this integral is exactly four times the area of the upper half-disk D, which is $\pi(\sqrt{2})^2/2 = 4\pi$. Thus $\int_C (-3y - e^{x^2})dx + (x - e^{y^2})dy = 8\pi$.

11 Outward flux is $\iint_S \vec{F} \cdot n dS$, which, by the divergence theorem, is equal to $\iiint_E (\nabla \cdot F)dV$

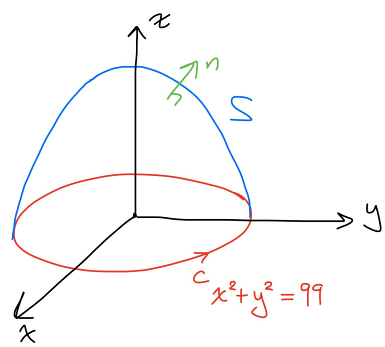
$\nabla \cdot \vec{F} = -1 - 1 - 1 = -3$

Therefore our outward flux is $\iiint_E 3dV$, which equals 3 times the volume of our cube, which is 2^3.

Thus Outward flux is equal to $-3 \cdot 8 = -24$.



12 The boundaryof the hyperboloid is the circle $x^2 + y^2 = 99$.



Stokes' Theorem tells us that

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds = \oint_C \vec{F}\, d\vec{r}$$

We can parametrize C by $\vec{r}(t) =< \sqrt{99}\cos(t), \sqrt{99}\sin(t), 0 >$, $0 \leq t < 2\pi$.
Then $\vec{r}\,'(t) =< -\sqrt{99}\sin(t), \sqrt{99}\cos(t), 0 >$.

$$\oint_C \vec{F}\, d\vec{r} = \int_0^{2\pi} < 0, \sqrt{99}\cos(t), \sqrt{99}\sin(t) > \cdot < -\sqrt{99}\sin(t), \sqrt{99}\cos(t), 0 > dt = 99 \int_0^{2\pi} \cos^2(t) dt$$

$$\frac{99}{2} \int_0^{2\pi} \cos(2t) + 1 dt = \frac{99}{2} \left(\frac{\sin(2t)}{2} + t \Big|_0^{2\pi} \right) = 99\pi$$