Math 42 | Tufts University | Monday, April 4th, 2016
Calculus III | Department of Mathematics | 12:00–1:20 pm

Exam 2 Solutions

1 (a) $\nabla w =< 2x, 2y, 2z >, \nabla w(1,1,1) =< 2,2,2 >$.

(b) In point-normal form, $2(x-1)+2(y-1)+2(z-1)=0$. In general form, $2x+2y+2z-6=0$.

2 $\nabla z =< e^{x+y}, e^{x+y} >$, so $dz = e^{x+y}dx + e^{x+y}dy$.

$e^{0+0} = 1$, $\Delta z \approx 1\Delta x + 1\Delta y = 0.1 - 0.05 = 0.05$.

3 First we find the critical values by setting $\nabla f =< ye^x, e^x - e^y >=< 0,0 >$.

$ye^x = 0 \implies y = 0 \implies e^x - 1 = 0 \implies x = 0$.

Taking the second derivatives, we have $D = \det \begin{bmatrix} ye^x & e^x \\ e^x & -e^y \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = -1 < 0$.

Therefore $(0,0)$ is a saddle point of f.

4 The distance from a point (x,y,z) on our cone to $(0,-1,0)$ is $d = \sqrt{(x-0)^2+(y+1)^2+(z-0)^2}$.

Approach One: Reducing Dimension.

Plug in the function $z = \sqrt{x^2-y^2}$ to d to get $d = \sqrt{(x-0)^2+(y+1)^2+(\sqrt{x^2+y^2}-0)^2}$.

We can look at $f(x,y) = d^2 = x^2+(y+1)^2+(x^2+y^2)$ and minimize that.

$\nabla f =< 4x, 4y+2 >=< 0,0 > \implies (x,y) = (0,-1/2)$.

Taking the second derivatives, we have $D = \det \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 16 > 0$.

Since $f_{xx} = 4 > 0$, the point $(0,-1/2)$ is the minimum of the function f and thus a relative minimum of the distance. From the geometry of the cone, it is clear that this relative minimum is an absolute minimum. Therefore the points $(0,-1/2,\pm 1/2)$ on the cone are closest to $(0,-1,0)$.

Approach Two: Lagrange multipliers.

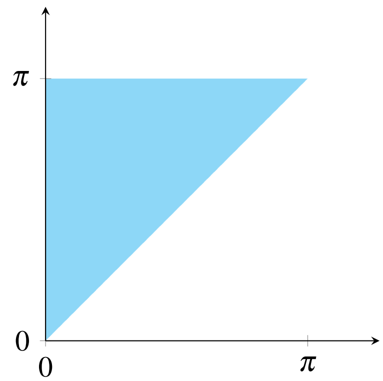
We can consider this to be the three variable function $f(x,y,z) = x^2+(y+1)^2+z^2$ constrained by the condition $g(x,y,z) = z^2-x^2-y^2 = 0$.

We must solve $\nabla f =< 2x, 2y+2, 2z >= \lambda < -2x, -2y, 2z >= \lambda \nabla g$

$$\begin{aligned} 2x &= -\lambda 2x \\ 2y+2 &= -\lambda 2y \\ 2z &= \lambda 2z \\ z^2-x^2-y^2 &= 0 \end{aligned} \quad \implies \quad \begin{aligned} 2x(1+\lambda) &= 0 \\ 2y(1+\lambda)+2 &= 0 \\ 2z(1-\lambda) &= 0 \\ z^2-x^2-y^2 &= 0 \end{aligned}$$

If $\lambda \neq 1$, then equation three tells us $z = 0$, and by equation four, we would have to have $x = 0$ and $y = 0$. However, if $y = 0$, the second equation would say $2 = 0$, which is false. Thus we must have $\lambda = 1$, which implies $x = 0$ and $y = -1/2$, which makes $z^2 = 0+(-1/2)^2 = 1/4$. From the geometry of the cone, it is clear that the point $(0,-1/2,\pm 1/2)$ minimizes $f(x,y,z)$, and thus is the point on the cone is closest to $(0,-1,0)$.

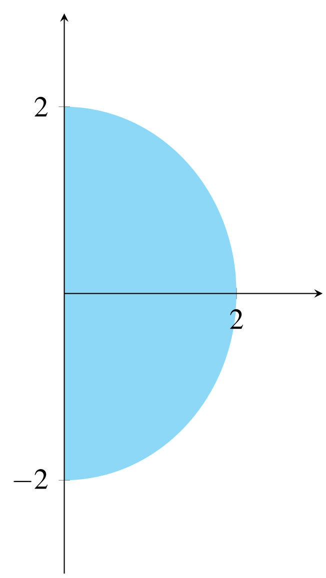
5 The region we are integrating over is the following:



Thus we can change our limits to be $0 \geq y \geq \pi, 0 \geq x \geq y$.

$$\int_0^{\pi}\int_x^{\pi} \sin(y^2)dydx = \int_0^{\pi}\int_0^{y} \sin(y^2)dxdy = \int_0^{\pi} y\sin(y^2)dy = -\frac{\cos(y^2)}{2}\bigg|_0^{\pi} = -\frac{\cos(\pi^2)}{2} + \frac{1}{2}$$

6 The region we are integrating over is the following:



In rectangular coordinates,

$$\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} 2xydxdy = \int_{-2}^{2}\left(x^2\Big|_0^{\sqrt{4-y^2}} y\right)dy = \int_{-2}^{2} y\left(\sqrt{4-y^2}^2\right)dy = \int_{-2}^{2} 4y - y^3dx$$

$$= 2y^2 - \frac{y^4}{4}\bigg|_{-2}^{2} = (8-4)-(8-4) = 0$$

In polar coordinates,

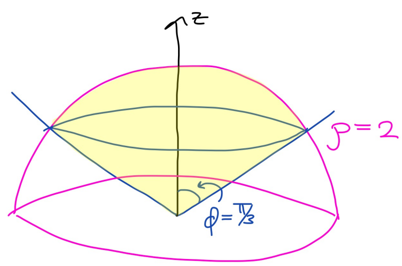
$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} 2xy\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_{0}^{2} 2(r\cos\theta)(r\sin\theta)r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_{0}^{2} 2r^3(\cos\theta)(\sin\theta)\,dr\,d\theta$$

$$\int_{-\pi/2}^{\pi/2}\left(\left.\frac{r^4}{2}\right|_0^2\right)(\cos\theta)(\sin\theta)\,d\theta = \int_{-\pi/2}^{\pi/2} 8(\cos\theta)(\sin\theta)\,d\theta = 4\sin^2\theta\big|_{-\pi/2}^{\pi/2} = 4-4=0$$

7 (a) $z=\sqrt{4-x^2-y^2} \implies x^2+y^2+z^2=4$, so in spherical, this is $\rho=2$.

(b) $z=\frac{1}{\sqrt{3}}\sqrt{x^2+y^2} \implies \frac{r}{z}=\sqrt{3} \implies \tan\varphi=\sqrt{3}$, so in spherical, this is $\varphi=\pi/3$.

(c) This is an 'ice-cream cone' shape. $\int_0^{2\pi}\int_0^{\pi/3}\int_0^2 1\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

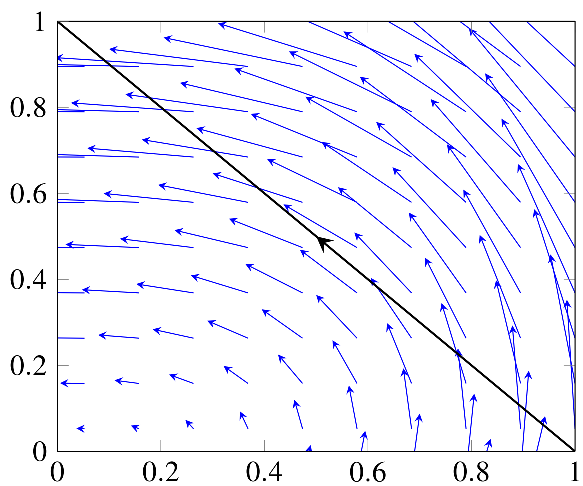


(d)

$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^2 1\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}\left.\frac{\rho^3}{3}\right|_0^2\sin\varphi\,d\varphi\,d\theta = \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/3}\sin\varphi\,d\varphi\,d\theta$$

$$= \frac{8}{3}\int_0^{2\pi} -\cos\varphi\big|_0^{\pi/3}\,d\theta = \frac{4}{3}\int_0^{2\pi} d\theta = \frac{8\pi}{3}$$

8 (a)



(b) $\vec{r}(t) = <1-t, t>, t\in[0,1]$ works.

(c) $\vec{r}'(t) =< -1, 1 >$, so $|\vec{r}'(t)| = \sqrt{2}$
$|\vec{F}(r(t))|^2 = y^2 + x^4 = t^2 + (1-t)^4$

$$\int_C |\vec{F}|^2 ds = \int_0^1 (t^2 + (1-t)^4)\sqrt{2}dt = \sqrt{2}\left(\frac{t^3}{3} - \frac{(1-t)^5}{5}\right)\Big|_0^1 = \frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{5} = \frac{8\sqrt{2}}{15}$$

(d)

$$\int_C \vec{F} \cdot \mathbf{T} ds = \int_0^1 \vec{F} \cdot \mathbf{r}'(t) dt = \int_0^1 < -t, (1-t)^2 > \cdot < -1, 1 > dt = \int_0^1 (t + (1-t)^2) dt$$

$$= \int_0^1 (t^2 - t + 1) dt = \frac{t^3}{3} - \frac{t^2}{2} + t\Big|_0^1 = \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}$$

(e)

$$\int_C \vec{F} \cdot \mathbf{n} ds = \int_0^1 < -t, (1-t)^2 > \cdot < 1, 1 > dt = \int_0^1 (-t + (1-t)^2) dt$$

$$= \int_0^1 (t^2 - 3t + 1) dt = \frac{t^3}{3} - \frac{3t^2}{2} + t\Big|_0^1 = \frac{1}{3} - \frac{3}{2} + 1 = -\frac{1}{6}$$