Math 42	Tufts University	Monday, Februrary 22, 2016
Calculus III	Department of Mathematics	12:00–1:20 pm
	Exam 1 Solutions	

1 True

$|u \times v| = 0$ means that u and v are parallel.
$u \cdot v = 0$ means that u and v are orthogonal.
The only way that they can be both parallel and orthogonal is if one of them is the zero vector.

Alternatively, $|u \times v| = |u||v| \sin\theta$, and $u \cdot v = |u||v| \cos\theta$.
$\cos\theta$ and $\sin\theta$ aren't zero at the same points, so it must be that either $|u| = 0$ or $|v| = 0$.

2 False, the area of the triangle ABC is $\frac{1}{2}|\vec{AB} \times \vec{AC}|$.

3 True, $proj_u(v)$ is the projection of v in the direction of u.

4 $u \cdot v = |u||v| \cos\theta$.
$u \cdot v = 6 - 4 + 0 = 2$
$|u| = \sqrt{4+1+4} = 3$
$|v| = \sqrt{9+16} = 5$
$\cos\theta = \frac{2}{15}$

5 (a) Set $r(t) = m(s)$ component wise.
We must solve the system of equations

$$\begin{aligned} 2t + 4 &= 2s \\ 6t &= s + 3 \end{aligned}$$

From the first equation we get $t = s - 2$.
Substituting this into the second equation, we have

$$\begin{aligned} 6(s-2) &= s + 3 \\ 6s - 12 &= s + 3 \\ 5s &= 15 \\ s &= 3 \end{aligned}$$

Plugging this into our equation for t, we have $t = 1$.
The two curves intersect at $r(1) = m(3) = < 6, 6 >$.

(b) The two curves hit the point of intersection at different times, $t = 1$ and $s = 3$ respectively, so the particles do not collide.

6 (a) We cannot divide by zero. $x^2 + y^2 + 1 > 0$ for all (x, y), so the domain is $\mathbb{R}^2$.

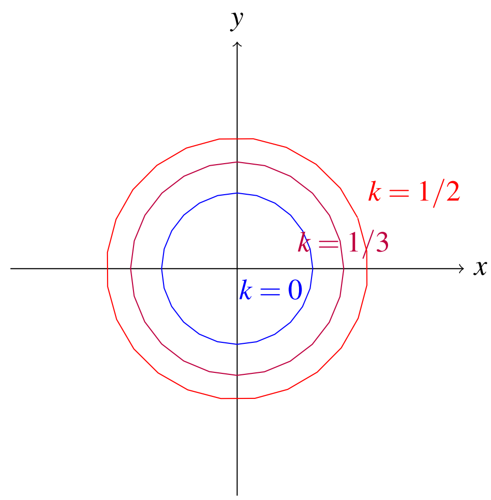
(b) To find the level curves we set $f(x, y) = k$.

$$\frac{x^2+y^2-1}{x^2+y^2+1}=k$$

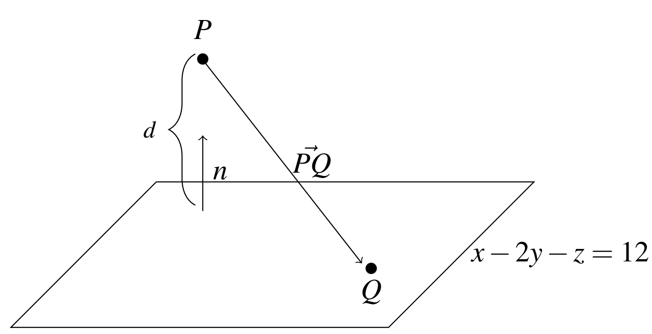
$$x^2+y^2-1=kx^2+ky^2+k$$
$$(1-k)x^2+(1-k)y^2=1+k$$
$$x^2+y^2=\frac{1+k}{1-k}$$

The level curves are circles centered at $(0,0)$ of radius $\sqrt{\frac{1+k}{1-k}}$ for $-1<k<1$ and a point for $k=-1$.

(c) If $k=0$, radius = 1, if $k=1/2$, radius $\sqrt{3}$, and if $k=1/3$, they have radius $\sqrt{2}$.



7 The vector normal to the plane is $n=<1,-2,-1>$. The vector $\vec{PQ}=<0,-9,-7>$.
We have the following picture:



The distance from the point to the plane is the magnutide $|proj_n(\vec{PQ})|$

$$proj_n(\vec{PQ})=\frac{\vec{PQ}\cdot n}{n\cdot n}n=\frac{0+18+7}{1+4+1}<1,-2,1>=\frac{25}{6}<1,-2,1>$$

$$|proj_n(\vec{PQ})|=\left|\frac{25}{6}<1,-2,1>\right|=\frac{25}{6}|<1,-2,1>|=\frac{25}{6}\sqrt{6}=\frac{25}{\sqrt{6}}$$

Equivalently, with scalar projection, we get

$$\left|scal_n(\vec{PQ})\right| = \left|\frac{\vec{PQ}\cdot n}{|n|}\right| = \left|\frac{0+18+7}{\sqrt{1+4+1}}\right| = \frac{25}{\sqrt{6}}$$

8 (a) We cannot take the logarithm of a nonpositive number, so we must have $x+y>0$.

(b) The first order partial derivatives are

$$f_x = \frac{e^y}{x+y}$$
$$f_y = e^y \ln(x+y) + \frac{e^y}{x+y}$$

(c) The second order partial derivatives are

$$f_{xx} = -\frac{e^y}{(x+y)^2}$$
$$f_{yy} = e^y \ln(x+y) + \frac{2e^y}{x+y} - \frac{e^y}{(x+y)^2}$$
$$f_{xy} = \frac{e^y}{x+y} - \frac{e^y}{(x+y)^2}$$
$$f_{yx} = \frac{e^y}{x+y} - \frac{e^y}{(x+y)^2}$$

9 The normal vector of Q is $n_1 =< -1,2,1 >$.
The normal vector of R is $n_2 =< 1,1,1 >$.
Since n_1 and n_2 are not equal, the two planes are not parallel, and hence intersect along a line. Their cross product is:

$$|n_1 \times n_2| = \det\begin{bmatrix} i & j & k \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = (2-1)i - (-1-1)j + (-1-2)k =< 1,2,-3 >$$

The intersection of the two planes is parametrized by

$$c(t) =< 0,1,-1 > +t < 1,2,-3 >=< t, 1+2t, -1-3t >$$

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10 The velocity of the curve is $r'(t) =< 9t^{7/2}, 3t^2 >$.
The speed of the curve is then $|r'(t)| = \sqrt{81t^7+9t^4} = 3t^2\sqrt{9t^3+1}$, and arclength is

$$S = \int_0^1 3t^2\sqrt{9t^3+1}dt$$
$$u = 9t^3+1,\ du = 27t^2dt$$
$$S = \frac{1}{9}\int_1^{10}\sqrt{u}du = \frac{1}{9}\left(\frac{2}{3}u^{3/2}\Big|_1^{10}\right) = \frac{2}{27}\left((10)^{3/2}-1\right) = \frac{2}{27}\left(\sqrt{1000}-1\right)$$