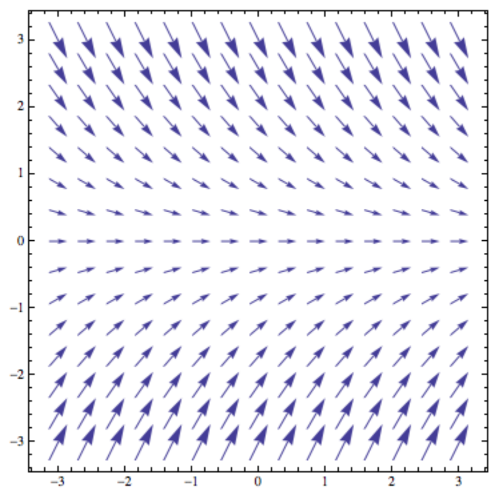
Mathematics 42	Tufts University	Monday, April 6, 2015
Calculus 3	Department of Mathematics	12:00–1:20pm
	Exam II	

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

(1) (12 points) Find the gradient of the function $f(x,y) = 3x - y^2$, and sketch the gradient field of f.

Solution: Since $f_x = 3$ and $f_y = -2y$, the gradient of f is $\nabla f(x,y) = \langle 3, -2y \rangle$.



(2) (12 points) Use cylindrical coordinates to find the volume of the solid above the cone $z = r$ and below the surface $z^2 = r$.

Solution: Our solid region D can be described as $\{(r, \theta, z) : z^2 \leq r \leq z, 0 \leq z \leq 1\}$, so its volume is given by

$$\iiint_D dV = \int_0^1 \int_0^{2\pi} \int_{z^2}^{z} r\,dr\,d\theta\,dz = 2\pi \int_0^1 \left(\frac{z^2 - z^4}{2}\right) dz = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$$

(3) (12 points) If C is the circle of radius 1 centered at the origin and oriented counterclockwise, and $\mathbf{F}(x,y) = \langle -y^3 + e^{3\cos(x)}, x^3 + \sin(y^3) \rangle$, use Green's Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Solution: Let $R = \{(x,y) : 0 \leq x^2 + y^2 \leq 1\}$. Since $\frac{\partial}{\partial x}(x^3 + \sin(y^3)) = 3x^2$ and $\frac{\partial}{\partial y}(-y^3 + e^{3\cos(x)}) = -3y^2$, Green's Theorem implies that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (3x^2 + 3y^2)\,dA = 3\int_0^{2\pi} \int_0^1 r^2 (r\,dr\,d\theta) = 6\pi \int_0^1 r^3\,dr = \frac{3\pi}{2}$$

(4) (14 points) Evaluate the line integral

$$\int_C (x^2 + y^2) ds$$

where C is the curve parametrized by $\mathbf{r}(t) = \langle 4e^{t/2}, 2e^{t/2}, 2t \rangle,\ \ 0 \le t \le \ln(\frac{12}{5})$.

Solution: Since $\mathbf{r}'(t) = \langle 2e^{t/2}, e^{t/2}, 2 \rangle$, we have that

$$ds = \|\mathbf{r}'(t)\| dt = \sqrt{4e^t + e^t + 4}\ dt = \sqrt{5e^t + 4}\ dt$$

Using the substitution $u = 5e^t + 4$, $du = 5e^t dt$, we see that our integral is

$$\int_C (x^2 + y^2) ds = 20 \int_0^{\ln(\frac{12}{5})} e^t \sqrt{5e^t + 4}\ dt = 4 \int_9^{16} \sqrt{u}\ du = \frac{8}{3}(16^{3/2} - 9^{3/2}) = \frac{296}{3}$$

(5) (12 points) Show that the vector field $\mathbf{F}(x, y, z) = \langle 3e^{3x+4y+5z}, 4e^{3x+4y+5z}, 5e^{3x+4y+5z} \rangle$ is conservative, and use this to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C has the parametrization

$$\mathbf{r}(t) = \langle \cos t, e^t \sin t, \cos 2t \rangle,\ \ 0 \le t \le \pi.$$

Solution: The vector field $\mathbf{F}$ is conservative since $\mathbf{F}(x, y, z) = \nabla \phi(x, y, z)$ where $\phi(x, y, z) = e^{3x+4y+5z}$. Applying the Fundamental Theorem of Calculus for line integrals, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla \phi \cdot d\mathbf{r} = \phi(\mathbf{r}(\pi)) - \phi(\mathbf{r}(0)) = \phi(-1, 0, 1) - \phi(1, 0, 1) = e^2 - e^8$$

(6) (12 points) Use spherical coordinates to evaluate

$$\iiint_D \frac{xyz}{(x^2 + y^2 + z^2)^{3/2}} dV$$

where $D = \{(x, y, z) : 0 \le x \le y,\ 4 \le x^2 + y^2 + z^2 \le 9\}$.

Solution: The region D can be described in spherical coordinates as

$$D = \{(\rho, \theta, \phi) : 2 \le \rho \le 3, \frac{\pi}{4} \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \pi\}$$

The value of our integral is then

$$\int_{\pi/4}^{\pi/2} \int_0^{\pi} \int_2^3 \left(\frac{(\rho \sin(\phi) \cos(\theta))(\rho \sin(\phi) \sin(\theta))(\rho \cos(\phi))}{\rho^3} \right) (\rho^2 \sin(\phi) d\rho d\phi d\theta)$$

$$= \int_{\pi/4}^{\pi/2} \int_0^{\pi} \int_2^3 \rho^2 \sin^3(\phi) \cos(\phi) \sin(\theta) \cos(\theta) d\rho d\phi d\theta$$

$$= \left(\int_{\pi/4}^{\pi/2} \sin(\theta) \cos(\theta) d\theta \right) \left(\int_0^{\pi} \sin^3(\phi) \cos(\phi) d\phi \right) \left(\int_2^3 \rho^2 d\rho \right) = \left(\int_{\pi/4}^{\pi/2} \sin(\theta) \cos(\theta) d\theta \right) \cdot 0 \cdot \left(\int_2^3 \rho^2 d\rho \right) = 0$$

(7) (12 points) Write $\iiint_D f(x,y,z)dV$ as an iterated integral (but do not evaluate!) in each of the orders $dz\ dy\ dx$ and $dx\ dz\ dy$ if

$$D = \{(x,y,z) : -2 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 4-y\}$$

Solution: Based on the given description of D, we have that

$$\iiint_D f(x,y,z)dV = \int_{-2}^{2} \int_{x^2}^{4} \int_{0}^{4-y} f(x,y,z)\ dz\ dy\ dx$$

We can also describe D as

$$D = \{(x,y,z) : 0 \le y \le 4, 0 \le z \le 4-y, -\sqrt{y} \le x \le \sqrt{y}\}$$

so the integral can also be written as $\int_0^4 \int_0^{4-y} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\ dx\ dz\ dy$.

(8) (14 points) Use Lagrange multipliers to find the maximum and minimum values of $f(x,y,z) = x+y-z$ subject to the constraint $x^2+y^2+z^2 = 1$.

Solution: Let $g(x,y,z) = x^2+y^2+z^2-1$. The system $\nabla f(x,y,z) = \lambda \cdot \nabla g(x,y,z), g(x,y,z) = 0$ can be written as

$$\begin{aligned} 1 &= 2\lambda x \\ 1 &= 2\lambda y \\ -1 &= 2\lambda z \\ x^2+y^2+z^2 &= 1 \end{aligned}$$

Any of the first three equations imply that $\lambda \neq 0$, so we have that $x = y = \frac{1}{2\lambda}$ and $z = -\frac{1}{2\lambda}$. Therefore $\frac{3}{4\lambda^2} = 1$, i.e. $\lambda = \pm\frac{\sqrt{3}}{2}$. It follows that the constrained extreme values for our problem are $f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \sqrt{3}$ (the maximum) and $f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = -\sqrt{3}$ (the minimum).