1. (10 points) **True or False - No Partial Credit**: On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) \((i + 2j + 3k) \cdot (3i - 2j + 6k) = 3i - 4j + 18k\).
   **Answer:** False (dot product should be scalar)

   (b) If \(u \times v\) is parallel to \(w\), then \(u\) is parallel to \(v \times w\).
   **Answer:** False

   (c) The line \(r(t) = \langle 2 + t, 3 - 2t, 5t \rangle\) is orthogonal to the plane \(x - 2y + 5z = 4\).
   **Answer:** True

   (d) \(1 - z^2 = 2x^2 + 4y^2\) describes an ellipsoid.
   **Answer:** True

2. (5 points) Find the vector projection of \(v = j + 4k\), on the vector \(u = i + 4j + 8k\).
   **Answer:**
   \[
   \text{proj}_u v = \frac{v \cdot u}{|u|^2} u \\
   \Rightarrow u \cdot v = 4 + 32 = 36. \\
   |u| = \sqrt{1 + 16 + 64} = 9. \\
   \Rightarrow \text{proj}_u v = \frac{4}{5}\langle 1, 4, 8 \rangle.
   \]

3. (10 points) Let \(r_1(t) = \langle 2 - t, 3 + t, 1 + 2t \rangle\) and \(r_2(s) = \langle 4 + 2s, 2 - s, 4 + 3s \rangle\) be two lines.

   (a) These two lines intersect at a point. Find the coordinates of this point.
   **Answer:** Setting the \(x\) values equal:
   \[2 - t = 4 + 2s \Rightarrow t = -2 - 2s.\]

   Setting \(y\) values equal:
   \[3 + t = 2 - s \Rightarrow s = -1 - t.\]

   Combining the two equations give \(t = -2 + 2 + 2t \Rightarrow t = 0 \Rightarrow s = -1.\)

   \[\Rightarrow r_1(0) = r_2(-1) = \langle 2, 3, 1 \rangle\]

   The coordinate of intersection is \(\langle 2, 3, 1 \rangle\).
(b) Find an equation of the plane containing the two lines, \( \mathbf{r}_1(t) \) and \( \mathbf{r}_2(s) \).

**Answer:** Since both lines are in the plane, their parallel vectors, \( \langle -1, 1, 2 \rangle \) and \( \langle 2, -1, 3 \rangle \), respectively, must both be perpendicular to the normal vector of the plane. Thus,

\[
\mathbf{n} = \langle -1, 1, 2 \rangle \times \langle 2, -1, 3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 2 & -1 & 3 \end{vmatrix} = (5, 7, -1).
\]

With the point of intersection, \((2, 3, 1)\) being in the plane we get the equation:

\[
5(x - 2) + 7(y - 3) - (z - 1) = 0 \Rightarrow 5x + 7y - z = 30.
\]

4. (10 points) Tom Brady is going to defy gravity and throw a “perfect spiral.” Let \( \mathbf{r}(t) \) describe the path of a deflated football traveling through space, such that at time \( t = 0 \), the ball is at position \((0, 0, 2)\), moving with velocity, \( \mathbf{v}(t) = 20\mathbf{i} + \cos(t)\mathbf{j} - \sin(t)\mathbf{k} \).

(a) Find the position and the acceleration of the ball at any given time \( t \).

**Answer:**

\[
\mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \langle 20t + c_1, \sin(t) + c_2, \cos(t) + c_3 \rangle.
\]

\( \Rightarrow \mathbf{r}(0) = \langle 0, 0, 2 \rangle = \langle c_1, c_2, 1 + c_3 \rangle. \)

\( \Rightarrow \mathbf{r}(t) = \langle 20t, \sin(t), \cos(t) + 1 \rangle. \)

\( \mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, -\sin(t), -\cos(t) \rangle. \)

(b) Find the length of the arc traveled by the ball from \( t = 0 \) to \( t = 5 \).

**Answer:**

\[
L = \int_{0}^{5} |\mathbf{v}(t)| \, dt = \int_{0}^{5} \sqrt{20^2 + \cos^2(t) + \sin^2(t)} \, dt
\]

\[
= \int_{0}^{5} \sqrt{401} \, dt = \sqrt{401} t_{0}^{5} = 5\sqrt{401} \rightarrow \text{Touchdown!}
\]
5. (15 points) Let \( y = -x^2 - z^2 \). Draw the following four sketches:

(a) Sketch the intersection of the surface with the plane \( y = -1 \).
   
   **Answer:**
   \[
   -1 = -x^2 - z^2 \Rightarrow x^2 + z^2 = 1
   \]
   It’s the unit circle

(b) Sketch the intersection of the surface with the plane \( x = 0 \).
   
   **Answer:**
   \[
   y = -z^2
   \]
   It’s a parabola!

(c) Sketch the intersection of the surface with the plane \( z = 0 \).
   
   **Answer:**
   \[
   y = -x^2
   \]
   It’s a parabola!
(d) Sketch the surface given by this equation.

Answer:

6. (15 points) Assume that $z$ is an implicitly defined function of $x$ and $y$ determined by the surface equation $x^2y + yz^2 - e^{xz} = 3$.

(a) Find $\frac{\partial z}{\partial x}$ at $(2, 1, 0)$.

Answer: Take the partial derivative with $x$ of the entire equation:

$$\frac{\partial}{\partial x} (x^2y + yz^2 - e^{xz} = 3) = 2xy + 2yzz_x - e^{xz} (z + xz_x) = 0$$

$$\Rightarrow z_x = \frac{ze^{xz} - 2xy}{2yz - xe^{xz}}$$

$$\frac{\partial z}{\partial x}(2, 1, 0) = \frac{4}{-2} = -2$$

(b) Find $\frac{\partial z}{\partial y}$ at $(2, 1, 0)$. Answer: Take the partial derivative with $y$ of the entire equation:

$$\frac{\partial}{\partial y} (x^2y + yz^2 - e^{xz} = 3) = x^2 + z^2 + 2yzz_y - e^{xz}xz_y = 0$$

$$\Rightarrow z_y = \frac{-x^2 - z^2}{2yz - xe^{xz}}$$

$$\frac{\partial z}{\partial y}(2, 1, 0) = \frac{4}{-2} = -2$$
7. (10 points) Let \( f(x, y) \) be a function of two variables such that \( f(2, -1) = 6 \) and \( \nabla f(2, -1) = (3, -4) \).

a. Write an equation of the tangent plane to the graph of \( f \) at the point \( (2, -1, 6) \).
   **Answer:**
   \[ z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \]
   \[ \Rightarrow z = 6 + 3(x - 2) - 4(y + 1) \Rightarrow z = 3x - 4y - 4. \]

b. Estimate \( f(2.1, -0.9) \) using linear approximation or differentials.
   **Answer:** The linearization is the same as the tangent plane, \( L(x, y) = z \):
   \[ \Rightarrow L(2.1, -0.9) = 6 + 3(2.1) - 4(-0.9) - 4 = 6 + 6.3 - 4 = 5.3. \]

8. (10 points) Consider the function \( f(x, y) = 2x^3 - y^4 \).

   (a) Find a unit vector in the direction in which \( f \) is decreasing the fastest at the point \( (1, 1) \).
   **Answer:** This occurs in the negative direction of the gradient.
   \[ \nabla f = (6x^2, -4x^3), \]
   \[ \nabla f(1, 1) = (6, -4). \]
   \[ |\nabla f(1, 1)| = \sqrt{36 + 16} = \sqrt{52}. \]
   \[ -\frac{\nabla f(1, 1)}{|\nabla f(1, 1)|} = \left( \frac{-6}{\sqrt{52}}, \frac{4}{\sqrt{52}} \right). \]

   (b) Find the directional derivative of \( f \) in the direction of the vector \( v = 3\hat{i} + 4\hat{j} \) at the point \( (1, 1) \).
   **Answer:** Direction is \( \hat{v} = \frac{1}{\sqrt{9 + 16}} (3, 4) = \left( \frac{3}{5}, \frac{4}{5} \right) \).
   \[ D_{\hat{v}}f(1, 1) = \hat{v} \cdot \nabla f(1, 1) = \frac{18}{5} - \frac{16}{5} = \frac{2}{5}. \]

9. (15 points) Let \( f(x, y) = 9x - 3x^3 - 12y + y^3 \).

   (a) Find all critical points of \( f(x, y) \) in the \( xy \)-plane.
   **Answer:** The critical points occur when the partial derivatives are 0 or undefined.
   \[ f_x = 9 - 9x^2 \] and \( f_y = 3y^2 - 12. \)
   These are never undefined and:
   \[ 9 - 9x^2 = 0 \quad \Rightarrow x^2 = 1 \quad \Rightarrow x = \pm 1 \]
   \[ 3y^2 - 12 = 0 \quad \Rightarrow y^2 = 4 \quad \Rightarrow y = \pm 2 \]
   Thus, the critical points are: \( (1, 2), (-1, 2), (1, -2) \), and \( (-1, -2) \).
(b) Determine where $f(x, y)$ has a local maximum, local minimum, or a saddle point. Determine the values of $f$ at these points as well.

**Answer:** Here, we use the second derivative test:

$$f_{xx} = -18x, \quad f_{yy} = 6y, \quad f_{xy} = 0, \quad D = (-18x)(6y) - 0^2 = -108xy.$$

- **$D(1, 2) < 0 \Rightarrow (1, 2)$ is a saddle with $f(1, 2) = -10$.**
- **$D(-1, 2) > 0, f_{xx}(-1, 2) > 0 \Rightarrow (-1, 2)$ is a local min with $f(-1, 2) = -22$.**
- **$D(1, -2) > 0, f_{xx} < 0 \Rightarrow (1, -2)$ is a local max with $f(1, -2) = 22$.**
- **$D(-1, -2) < 0 \Rightarrow (-1, -2)$ is a saddle with $f(-1, -2) = 10$.**

End of Exam