Tufts University

Math 42	**Department of Mathematics**	**February 23, 2015**
	Exam 2	**12:00 pm to 1:20 pm**

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (10 points) **True or False - No Partial Credit**: On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) $(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\cdot(3\mathbf{i}-2\mathbf{j}+6\mathbf{k}) = 3\mathbf{i}-4\mathbf{j}+18\mathbf{k}$.
   Answer: $\boxed{\text{False}}$ (dot product should be scalar)

   (b) If $\mathbf{u}\times\mathbf{v}$ is parallel to $\mathbf{w}$, then $\mathbf{u}$ is parallel to $\mathbf{v}\times\mathbf{w}$.
   Answer: $\boxed{\text{False}}$

   (c) The line $\mathbf{r}(t) = \langle 2+t, 3-2t, 5t\rangle$ is orthogonal to the plane $x-2y+5z=4$.
   Answer: $\boxed{\text{True}}$

   (d) $1-z^2 = 2x^2+4y^2$ describes an ellipsoid.
   Answer: $\boxed{\text{True}}$

   $\nabla F(x,y,z)$ is orthogonal to the surface $F(x,y,z)=0$.
   Answer: $\boxed{\text{True}}$

2. (5 points) Find the vector projection of $\mathbf{v}=\mathbf{j}+4\mathbf{k}$, on the vector $\mathbf{u}=\mathbf{i}+4\mathbf{j}+8\mathbf{k}$.
   Answer:
   $$\mathbf{proj_u v} = \frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|}\frac{\mathbf{u}}{|\mathbf{u}|}$$
   $\Rightarrow \mathbf{u}\cdot\mathbf{v} = 4+32 = 36.$
   $|\mathbf{u}| = \sqrt{1+16+64} = 9.$
   $$\Rightarrow \mathbf{proj_u v} = \boxed{\tfrac{4}{9}\langle 1,4,8\rangle}.$$

3. (10 points) Let $\mathbf{r}_1(t) = \langle 2-t, 3+t, 1+2t\rangle$ and $\mathbf{r}_2(s) = \langle 4+2s, 2-s, 4+3s\rangle$ be two lines.

   (a) These two lines intersect at a point. Find the coordinates of this point.
   Answer: Setting the x values equal:
   $$2-t = 4+2s \Rightarrow t = -2-2s.$$
   Setting y values equal:
   $$3+t = 2-s \Rightarrow s = -1-t.$$
   Combining the two equations give $t = -2+2+2t \Rightarrow t=0 \Rightarrow s=-1$.
   $$\Rightarrow \mathbf{r}_1(0) = \mathbf{r}_2(-1) = \langle 2,3,1\rangle$$
   The coordinate of intersection is $\boxed{(2,3,1)}$.

(b) Find an equation of the plane containing the two lines, $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$.
Answer: Since both lines are in the plane, their parallel vectors, $\langle -1, 1, 2\rangle$ and $\langle 2, -1, 3\rangle$, respectively, must both be perpendicular to the normal vector of the plane. Thus,

$$\mathbf{n} = \langle -1, 1, 2\rangle \times \langle 2, -1, 3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 2 & -1 & 3 \end{vmatrix} = \langle 5, 7 - 1\rangle.$$

With the point of intersection, $(2, 3, 1)$ being in the plane we get the equation:

$$5(x-2) + 7(y-3) - (z-1) = 0 \Rightarrow \boxed{5x + 7y - z = 30}.$$

4. (10 points) Tom Brady is going to defy gravity and throw a "perfect spiral." Let $\mathbf{r}(t)$ describe the path of a deflated football traveling through space, such that at time $t = 0$, the ball is at position $(0, 0, 2)$, moving with velocity, $\mathbf{v}(t) = 20\mathbf{i} + \cos(t)\mathbf{j} - \sin(t)\mathbf{k}$.

(a) Find the position and the acceleration of the ball at any given time t.
Answer: $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \langle 20t + c_1, \sin(t) + c_2, \cos(t) + c_3\rangle$.
$\Rightarrow \mathbf{r}(0) = \langle 0, 0, 2\rangle = \langle c_1, c_2, 1 + c_3\rangle$. $\Rightarrow \boxed{\mathbf{r}(t) = \langle 20t, \sin(t), \cos(t) + 1\rangle}$.
$\mathbf{a}(t) = \mathbf{v}'(t) = \boxed{\mathbf{a}(t) = \langle 0, -\sin(t), -\cos(t)\rangle}$.

(b) Find the length of the arc traveled by the ball from $t = 0$ to $t = 5$.
Answer:

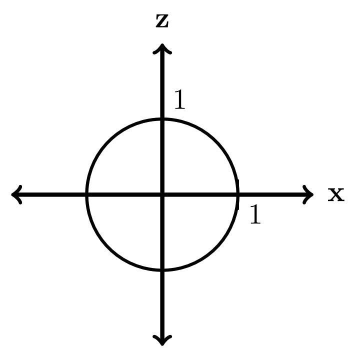
$$L = \int_0^5 |\mathbf{v}(t)|\,dt = \int_0^5 \sqrt{20^2 + \cos^2(t) + \sin^2(t)}\,dt$$
$$= \int_0^5 \sqrt{401}\,dt = \sqrt{401}t\Big|_0^5 = \boxed{5\sqrt{401} \to \text{Touchdown!}}.$$

5. (15 points) Let $y = -x^2 - z^2$. Draw the following four sketches:

(a) Sketch the intersection of the surface with the plane $y = -1$.
Answer:

$$-1 = -x^2 - z^2 \Rightarrow x^2 + z^2 = 1$$

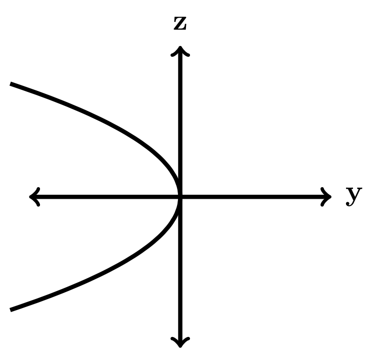
It's the unit circle



(b) Sketch the intersection of the surface with the plane $x = 0$.
Answer:

$$y = -z^2$$

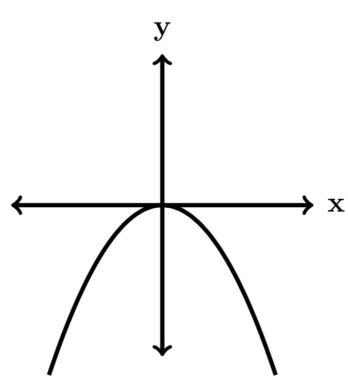
It's a parabola!



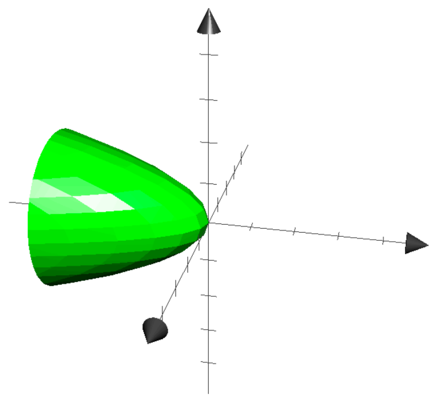
(c) Sketch the intersection of the surface with the plane $z = 0$.
Answer:

$$y = -x^2$$

It's a parabola!



(d) Sketch the surface given by this equation.
Answer:



6. (15 points) Assume that z is an implicitly defined function of x and y determined by the surface equation $x^2y + yz^2 - e^{xz} = 3$.

(a) Find $\frac{\partial z}{\partial x}$ at $(2, 1, 0)$.
Answer: Take the partial derivative with x of the entire equation:

$$\frac{\partial}{\partial x}\left(x^2y + yz^2 - e^{xz} = 3\right) = 2xy + 2yzz_x - e^{xz}\left(z + xz_x\right) = 0$$

$$\Rightarrow z_x = \frac{ze^{xz} - 2xy}{2yz - xe^{xz}}$$

$$\boxed{\frac{\partial z}{\partial x}(2, 1, 0) = \frac{-4}{-2} = 2}.$$

(b) Find $\frac{\partial z}{\partial y}$ at $(2, 1, 0)$. **Answer:** Take the partial derivative with y of the entire equation:

$$\frac{\partial}{\partial y}\left(x^2y + yz^2 - e^{xz} = 3\right) = x^2 + z^2 + 2yzz_y - e^{xz}xz_y = 0$$

$$\Rightarrow z_y = \frac{-x^2 - z^2}{2yz - xe^{xz}}$$

$$\boxed{\frac{\partial z}{\partial y}(2, 1, 0) = \frac{-4}{-2} = 2}.$$

7. (10 points) Let $f(x, y)$ be a function of two variables such that $f(2, -1) = 6$ and $\nabla f(2, -1) = \langle 3, -4 \rangle$.

   a. Write an equation of the tangent plane to the graph of f at the point $(2, -1, 6)$.
   Answer:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
$$\Rightarrow z = 6 + 3(x - 2) - 4(y + 1) \Rightarrow \boxed{z = 3x - 4y - 4}.$$

   b. Estimate $f(2.1, -0.9)$ using linear approximation or differentials.
   Answer: The linearization is the same as the tangent plane, $L(x, y) = z$:

$$\Rightarrow L(2.1, -0.9) = 3(2.1) - 4(-0.9) - 4 = 6.3 + 3.6 - 4 = \boxed{5.9}.$$

8. (10 points) Consider the function $f(x, y) = 2x^3 - y^4$.

   (a) Find a unit vector in the direction in which f is *decreasing* the fastest at the point $(1, 1)$.
   Answer: This occurs in the negative direction of the gradient.

$$\nabla f = \langle 6x^2, -4x^3 \rangle,$$
$$\nabla f(1, 1) = \langle 6, -4 \rangle.$$
$$|\nabla f(1, 1) = \sqrt{36 + 16} = \sqrt{52}.$$

$-\frac{\nabla f(1,1)}{|\nabla f(1,1)|} = \boxed{\langle \frac{-6}{\sqrt{52}}, \frac{4}{\sqrt{52}} \rangle}.$

   (b) Find the directional derivative of f in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ at the point $(1, 1)$.
   Answer: Direction is $\hat{v} = \frac{1}{\sqrt{9+16}} \langle 3, 4 \rangle = \langle \frac{3}{5}, \frac{4}{5} \rangle$.

$$D_{\hat{v}} f(1, 1) = \hat{v} \cdot \nabla f(1, 1) = \frac{18}{5} - \frac{16}{5} = \boxed{\frac{2}{5}}.$$

9. (15 points) Let $f(x, y) = 9x - 3x^3 - 12y + y^3$.

   (a) Find all critical points of $f(x, y)$ in the $xy-$plane.
   Answer: The critical points occur when the partial derivatives are 0 or undefined.

$$f_x = 9 - 9x^2 \text{ and } f_y = 3y^2 - 12.$$

   These are never undefined and:

$$\begin{aligned} f_x &= 0 \quad \text{and} & f_y &= 0 \\ 9 - 9x^2 &= 0 & 3y^2 - 12 &= 0 \\ x^2 &= 1 & y^2 &= 4 \\ x &= \pm 1 & y &= \pm 2 \end{aligned}$$

   Thus, the critical points are: $\boxed{(1, 2)}$, $\boxed{(-1, 2)}$, $\boxed{(1, -2)}$, and $\boxed{(-1, -2)}$.

(b) Determine where $f(x,y)$ has a local maximum, local minimum, or a saddle point. Determine the values of f at these points as well.
Answer: Here, we use the second derivative test:

$$f_{xx} = -18x, \quad f_{yy} = 6y, \quad f_{xy} = 0, \quad D = (-18x)(6y) - 0^2 = -108xy.$$

$\boxed{D(1,2) < 0 \Rightarrow (1,2) \text{ is a saddle with } f(1,2) = -10.}$

$\boxed{D(-1,2) > 0, f_{xx}(-1,2) > 0 \Rightarrow (-1,2) \text{ is a local min with } f(-1,2) = -22.}$

$\boxed{D(1,-2) > 0, f_{xx} < 0 \Rightarrow (1,-2) \text{ is a local max with } f(1,-2) = 22.}$

$\boxed{D(-1,-2) < 0 \Rightarrow (-1,-2) \text{ is a saddle with } f(-1,-2) = 10.}$

End of Exam