Math 42 | Tufts University
Department of Mathematics
Final Exam Solutions | Spring 2013

1. (10 points) **True or False – no partial credit.**

   (a) The line $x = 2 - t,\ y = 1 + 3t,\ z = 1 + t$ is perpendicular to the plane $2x - 6y - 2z = 7$.
   True: The line has is parallel to the vector $\mathbf{v} = \langle -1, 3, 1\rangle$, and the plane has normal vector $\mathbf{n} = \langle 2, -6, -2\rangle = -2\,\mathbf{v}$.

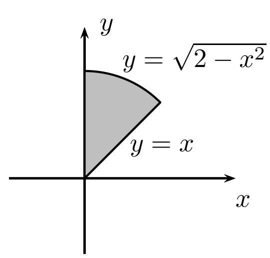
   (b) The vector $\mathbf{v} = 4\,\mathbf{i} - 2\,\mathbf{j} - \mathbf{k}$ is normal to the surface $x^2 - y^2 z = 3$ at the point $(2, 1, 1)$.
   True: Let $F(x, y, z) = x^2 - y^2 z$. Then the normal vector to the level surface $F(x, y, z) = 3$ at the point $(2, 1, 1)$ is $\nabla F|_{(2,1,1)} = (2x\,\mathbf{i} - 2yz\,\mathbf{j} - y^2\,\mathbf{k})\big|_{(2,1,1)} = 4\,\mathbf{i} - 2\,\mathbf{j} - \mathbf{k}$.

   (c) The line integral $\displaystyle\int_C x\cos y\,dx - x^2 \sin y\,dy$ in $\mathbb{R}^2$ is independent of path.

   False: The line integral above can be written as $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = x\cos y\,\mathbf{i} - x^2 \sin y\,\mathbf{j}$. If the line integral is independent of path, then $\mathbf{F}(x, y)$ is conservative, which would imply that $\dfrac{\partial}{\partial x}(-x^2\,\sin y) = \dfrac{\partial}{\partial y}(x\,\cos y)$, or that $-2x\,\sin y = -x\,\sin y$, which is evidently not true.

   (d) $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} xy\,dy\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r^3\,\sin\theta\,\cos\theta\,dr\,d\theta.$

   True: The region of integration for both integrals is shown below:



   Thus

$$\int_0^1 \int_x^{\sqrt{2-x^2}} xy\,dy\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} (r\,\sin\theta)\,(r\cos\theta)\,r\,dr\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r^3\,\sin\theta\,\cos\theta\,dr\,d\theta$$

(e) If $\mathbf{F}(x, y, z) = (x^3y^2 + z^3)\,\mathbf{i} - x^2y^3\,\mathbf{j} + (x + y)\,\mathbf{k}$, then the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

over any oriented closed surface S is equal to 0.

True: A simple calculation shows that the divergence $\nabla \cdot \mathbf{F} = 0$. If S is any oriented closed surface (with outward unit normal) such that S is the boundary of a solid E in $\mathbb{R}^3$, then according to the Divergence Theorem, $\iint_S \mathbf{F} \cdot \mathbf{n} = \iiint_E \nabla \cdot \mathbf{F}\, dV = 0$.

2. (10 points) The temperature T at the point (x, y, z) in $\mathbb{R}^3$ is given by $T(x, y, z) = e^x\, y^2\, z$.

(a) Find the gradient ∇T of the temperature at the point $(0, -1, 2)$.

Solution:

$$\begin{aligned} \nabla T(x, y, z) &= \frac{\partial T}{\partial x}\,\mathbf{i} + \frac{\partial T}{\partial y}\,\mathbf{j} + \frac{\partial T}{\partial z}\,\mathbf{k} \\ &= e^x y^2 z\,\mathbf{i} + 2e^x yz\,\mathbf{j} + e^x y^2\,\mathbf{k} \end{aligned}$$

Hence

$$\nabla T(0, -1, 2) = 2\,\mathbf{i} - 4\,\mathbf{j} + \mathbf{k}.$$

(b) At the point $(0, -1, 2)$, find the direction towards which the temperature increases most rapidly. Express the direction as a unit vector.

Solution: At the point $(0, -1, 2)$, T increases most rapidly in the direction of $\nabla T(0, -1, 2)$. Thus the *unit* vector in the direction of which T increases most rapidly at $(0, -1, 2)$ is

$$\begin{aligned} \mathbf{u} &= \frac{\nabla T(0, -1, 2)}{\|\nabla T(0, -1, 2)\|} \\ &= \frac{2\,\mathbf{i} - 4\,\mathbf{j} + \mathbf{k}}{\sqrt{21}}. \end{aligned}$$

(c) A gnat is flying along a path such that at the instant it is at the point $(2, -2, 1)$, its velocity vector is $\mathbf{v} = 3\,\mathbf{i} + 2\mathbf{j} + 2\,\mathbf{k}$. What is the rate of change of temperature along its path at this instant?

Solution: If $\mathbf{r}(t)$ denotes the position vector of the gnat at time t, and if the gnat is at the point $(0, -1, 2)$ at time t_0, then $\mathbf{r}(t_0) = -\mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{r}'(t_0) = 3\,\mathbf{i} + 2\mathbf{j} + 2\,\mathbf{k}$. By the Chain Rule, the rate of change of temperature being experienced by the gnat at time t_0 is then

$$\begin{aligned} T'(t_0) &= \left.\frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}\right|_{t=t_0} \\ &= \left.\left(\frac{\partial T}{\partial x}\,\mathbf{i} + \frac{\partial T}{\partial y}\,\mathbf{j} + \frac{\partial T}{\partial z}\,\mathbf{k}\right)\right|_{(0,-1,2)} \cdot \left.\left(\frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}\right)\right|_{t=t_0} \\ &= \nabla T(0, -1, 2) \cdot \mathbf{r}'(t_0) \\ &= (2\,\mathbf{i} - 4\,\mathbf{j} + \mathbf{k}) \cdot (3\,\mathbf{i} + 2\mathbf{j} + 2\,\mathbf{k}) \\ &= 0. \end{aligned}$$

3. (10 points) Let $f(x,y) = xy - x^2y - xy^2$.

(a) The point $(\frac{1}{3}, \frac{1}{3})$ is a critical point of $f(x,y)$. Find the other *three* critical points of $f(x,y)$.

Solution: The critical points of $f(x,y)$ occur where the partials of f are zero:

$$\frac{\partial f}{\partial x} = 0 \implies y - 2xy - y^2 = y(1-2x-y) = 0$$
$$\implies y = 0 \text{ or } 2x + y = 1.$$

and

$$\frac{\partial f}{\partial y} = 0 \implies x - x^2 - 2xy = x(1-x-2y) = 0$$
$$\implies x = 0 \text{ or } x + 2y = 1.$$

This results in four separate cases, resulting in four separate critical points:

Equations	Critical Point
$y = 0$ and $x = 0$	(0,0)
$y = 0$ and $x + 2y = 1$	(1,0)
$2x + y = 1$ and $x = 0$	(0,1)
$2x + y = 1$ and $x + 2y = 1$	$(\frac{1}{3}, \frac{1}{3})$

(b) Determine whether the critical point $(\frac{1}{3}, \frac{1}{3})$ gives a local maximum, a local minimum, or a saddle point for $f(x,y)$.

Solution: We apply the second derivative test. The second partials are:

$$f_{xx} = -2y, \qquad f_{xy} = 1 - 2x - 2y, \qquad f_{yy} = -2x.$$

Hence $f_{xx}(\frac{1}{3}, \frac{1}{3}) = -\frac{2}{3}$, $f_{xy}(\frac{1}{3}, \frac{1}{3}) = -\frac{1}{3}$, and $f_{yy}(\frac{1}{3}, \frac{1}{3}) = -\frac{2}{3}$. If follows that at $(\frac{1}{3}, \frac{1}{3})$,

$$f_{xx}\, f_{yy} - f_{xy}^2 = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) - \left(-\frac{1}{3}\right)^2 = \frac{1}{3} > 0,$$

and $f_{xx}(\frac{1}{3}, \frac{1}{3}) = -\frac{2}{3} < 0$. Therefore $f(\frac{1}{3}, \frac{1}{3})$ is a *local maximum.*

4. (10 points) The method of Lagrange multipliers is to be used to maximize the function $f(x,y) = 4x + 2y$ on the ellipse $\dfrac{x^2}{2} + y^2 = 1$.

(a) Write down the system of three equations in x, y, and λ which you will need to find the point (x,y) on the ellipse which maximizes the function f.

Solution: Let $g(x, y) = \dfrac{x^2}{2} + y^2$. Then the Lagrange multiplier equations are

$$\begin{aligned} f_x &= \lambda\, g_x \\ f_y &= \lambda\, g_y \\ g(x, y) &= 1, \end{aligned}$$

which give

$$\begin{aligned} 4 &= \lambda x \\ 2 &= \lambda(2y) \\ \frac{x^2}{2} + y^2 &= 1. \end{aligned}$$

(b) Using your answer to Part (a), find the point (x, y) on the ellipse which maximizes the function f.

Solution: The first two equations in Part (a) show that none of x, y, or λ is zero. Thus $x = 4/\lambda$, $y = 1/\lambda$. Plugging these into the third equation, we obtain

$$\frac{8}{\lambda^2} + \frac{1}{\lambda^2} = 1.$$

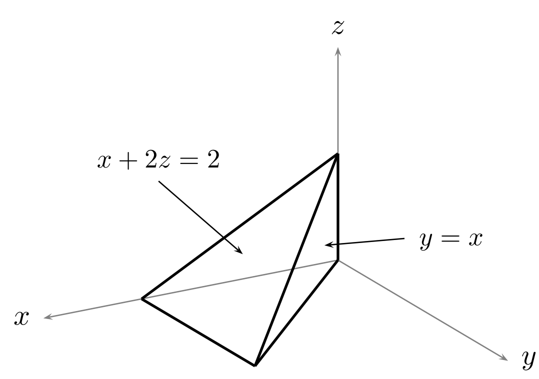
This results in $\lambda^2 = 9$, so $\lambda = \pm 3$. Hence $(x, y) = (\frac{4}{\lambda}, \frac{1}{\lambda}) = (\frac{4}{3}, \frac{1}{3})$ or $(x, y) = (-\frac{4}{3}, -\frac{1}{3})$. Now $f(-\frac{4}{3}, -\frac{1}{3}) = -6$ and $f(\frac{4}{3}, \frac{1}{3}) = 6$. We conclude that, on the ellipse, $f(x, y)$ is maximum at the point $(\frac{4}{3}, \frac{1}{3})$.

5. (10 points)

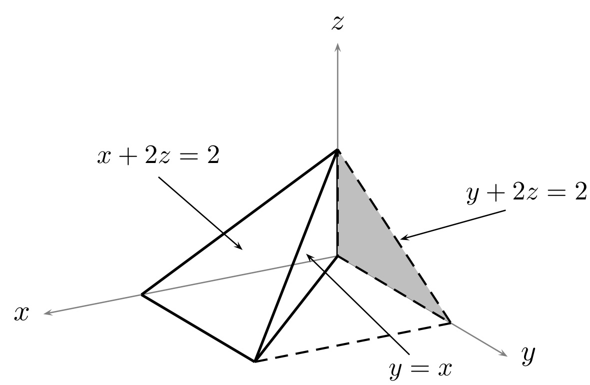
(a) The figure below shows the region of integration for the integral

$$\int_0^2 \int_0^x \int_0^{1-\frac{1}{2}x} f(x, y, z)\, dz\, dy\, dx.$$

Rewrite this integral as an equivalent iterated integral in the order $dx\, dz\, dy$.



Solution:



We use E to denote the solid of integration. Then, in the picture above, E projects onto the shaded triangular region in the (y, z)-plane. Thus

$$\int_0^2 \int_0^x \int_0^{1-\frac{1}{2}x} f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV$$
$$= \int_0^2 \int_0^{1-\frac{y}{2}} \int_y^{2-2z} f(x,y,z)\,dx\,dz\,dy.$$

(b) Express the triple integral

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} xyz^2\,dz\,dy\,dx$$

as an iterated triple integral in spherical coordinates. DO NOT EVALUATE.

Solution: The triple integral is taken over the solid E inside the sphere $x^2+y^2+z^2 = 4$ and above and below the first quadrant in the (x, y)-plane. Hence

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} xyz^2\,dz\,dy\,dx = \iiint_E xyz^2\,dV$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{\pi} \int_0^2 (\rho\,\sin\varphi\,\cos\theta)\,(\rho\,\sin\varphi\,\sin\theta)\,(\rho\,\cos\varphi)\,\rho^2\,\sin\varphi\,d\rho\,d\varphi\,d\theta$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{\pi} \int_0^2 \rho^5\,\sin^3\varphi\,\cos\varphi\,\sin\theta\,\cos\theta\,d\rho\,d\varphi\,d\theta$$

6. (10 points)

(a) Let $\mathbf{F}(x, y, z) = e^x yz\,\mathbf{i} + (e^x z + 2yz^2)\,\mathbf{j} + (e^x y + 2y^2 z + 3z^2)\,\mathbf{k}$. Find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

Solution: Even though it is assumed that $\mathbf{F}(x, y, z)$ is conservative, this fact can be easily verified by checking that $\nabla \times \mathbf{F} = \mathbf{0}$.

Now we are given that $\mathbf{F} = \nabla f(x, y, z) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$. Hence

$$\frac{\partial f}{\partial x} = e^x yz$$

$$\frac{\partial f}{\partial y} = e^x z + 2yz^2 \tag{1}$$

$$\frac{\partial f}{\partial z} = e^x y + 2y^2 z + 3z^2. \tag{2}$$

We integrate the both sides of the first equation above with respect to x to get

$$f(x, y, z) = e^x yz + A(y, z)$$

for some differentiable function $A(y, z)$. Hence by (1) we obtain

$$\frac{\partial f}{\partial y} = e^x z + \frac{\partial A}{\partial y} = e^x z + 2yz^2$$

and therefore $\frac{\partial A}{\partial y} = 2yz^2$. It follows that $A(y, z) = y^2 z^2 + B(z)$, so that

$$f(x, y, z) = e^x yz + y^2 z^2 + B(z).$$

Then taking the partial of both sides above with respect to z and taking (2) into account, we get

$$\frac{\partial f}{\partial z} = e^x y + 2y^2 z + B'(z) = e^x y + 2y^2 z + 3z^2.$$

Hence $B'(z) = 3z^2$, so that $B(z) = z^3 + C$. Hence $f(x, y, z) = e^x yz + y^2 z^2 + z^3 + C$. Since we just need one function, we can put $C = 0$, and have $f(x, y, z) = e^x yz + y^2 z^2 + z^3$.

(b) Evaluate the line integral

$$\int_C e^x yz\,dx + (e^x z + 2yz^2)\,dy + (e^x y + 2y^2 z + 3z^2)\,dz,$$

where C is the line segment from $(0, 0, 0)$ to $(2, 1, 1)$.

Solution: By the Fundamental Theorem of Line Integrals, we have

$$\begin{aligned}\int_C e^x yz\,dx + (e^x z + 2yz^2)\,dy + (e^x y + 2y^2 z + 3z^2)\,dz \\ &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \left(e^x yz + y^2 z^2 + z^3\right)\Big|_{(0,0,0)}^{(2,1,1)} \\ &= e^2 + 2.\end{aligned}$$

7. (10 points) Use Green's Theorem to evaluate the line integral

$$\oint_C (x^6 + xy)\,dx + x\,dy$$

where C is the closed curve consisting of the semicircle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$, followed by the line segment from $(-2, 0)$ to $(2, 0)$.

Solution: Let D be the region in the (x, y)-plane enclosed by the curve C. Then

$$\begin{aligned}\oint_C (x^6 + xy)\,dx + x\,dy &= \iint_D \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x^6 + xy)\right) dA \\ &= \iint_D (1 - x)\,dA \\ &= \iint_D dA - \iint_D x\,dA \\ &= \iint_D dA\end{aligned}$$

(since x is an odd function and D is symmetrical with respect to the y-axis)

$$\begin{aligned}&= \text{area}\,(D) \\ &= \frac{1}{2}\,\pi(2)^2 \\ &= 2\pi.\end{aligned}$$

8. (10 points) Consider the surface S (called the *helicoid*) given by the parametric equations

$$\begin{aligned}x &= u\,\cos v \\ y &= u\,\sin v \qquad\qquad (0 \leq u \leq 2,\ 0 \leq v \leq 2\pi) \\ z &= v\end{aligned}$$

Express the surface area of S as an iterated double integral in u and v. DO NOT EVALUATE.

Solution: The surface S can also be described by the vector valued function

$$\mathbf{r}(u, v) = u\,\cos v\,\mathbf{i} + u\,\sin v\,\mathbf{j} + v\,\mathbf{k}.$$

The surface area element is $dS = \|\mathbf{r}_u \times \mathbf{r}_v\|\,dudv$. Let us calculate $\mathbf{r}_u \times \mathbf{r}_v$. Now $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$ and $\mathbf{r}_v = -u\,\sin v\,\mathbf{i} + u\,\cos v\,\mathbf{j} + \mathbf{k}$. Hence

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\,\sin v & u\,\cos v & 1 \end{vmatrix} = \sin v\,\mathbf{i} - \cos v\,\mathbf{j} + u\,\mathbf{k}.$$

Thus $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{u^2+1}$, and therefore the surface area of the helicoid is

$$\begin{aligned}\iint_S dS &= \int_0^2 \int_0^{2\pi} \|\mathbf{r}_u \times \mathbf{r}_v\| \, du\, dv \\ &= \int_0^2 \int_0^{2\pi} \sqrt{u^2+1}\, du\, dv\end{aligned}$$

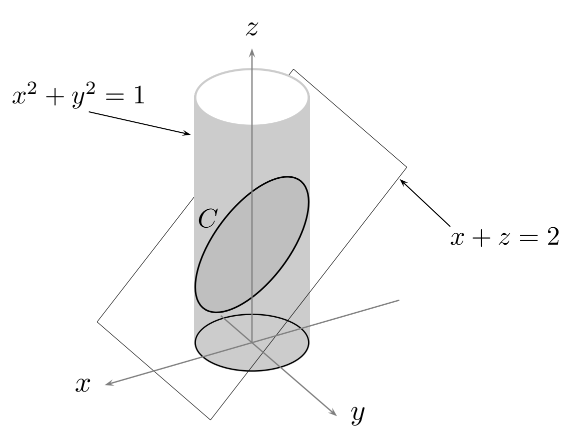
9. (10 points) Consider the vector field

$$\mathbf{F}(x,y,z) = xz^2\,\mathbf{i} + xy^2\,\mathbf{j} + x^2y\,\mathbf{k}.$$

(a) Find the curl $\nabla \times \mathbf{F}$ of the vector field $\mathbf{F}(x,y,z)$.
Solution:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^2 & xy^2 & x^2y \end{vmatrix} \\ &= x^2\,\mathbf{i} + (2xz - 2xy)\,\mathbf{j} + y^2\,\mathbf{k}.\end{aligned}$$

(b) Use Stokes' theorem to compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve of intersection of the plane $x+z=2$ and the cylinder $x^2+y^2=1$, oriented counterclockwise when viewed from above.



Solution: The ellipse C is the (oriented) boundary of the surface S, which is the region in the plane $z = 2 - x$ inside the cylinder $x^2+y^2=1$, with the upward orientation. The surface S projects onto the unit disk $D:\ x^2+y^2 \le 1$ in the (x,y)-plane. We write the components of the curl $\nabla \times \mathbf{F}$ found in Part (a) as $P(x,y,z) = x^2,\ Q(x,y,z) = 2xz - 2xy,$

and $R(x, y, z) = y^2$. Then by Stokes' Theorem,

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS \\
&= \iint_D \left(-P\,\frac{\partial z}{\partial x} - Q\,\frac{\partial z}{\partial y} + R \right)\, dx\, dy \\
&= \iint_D \left(-x^2(-1) - (2xz - 2xy)(0) + y^2 \right)\, dx\, dy \\
&= \iint_D (x^2 + y^2)\, dS \\
&= \int_0^{2\pi} \int_0^1 (r^2)\, r\, dr\, d\theta \\
&= \frac{\pi}{2}.
\end{aligned}$$

10. (10 points) Use the divergence theorem to compute the *outward* flux of the vector field

$$\mathbf{F}(x, y, z) = 3xz^2\,\mathbf{i} + y^3\,\mathbf{j} + 3x^2z\,\mathbf{k}$$

across the sphere $x^2 + y^2 + z^2 = 4$.

Solution: Let S be the sphere $x^2 + y^2 + z^2 = 4$, oriented outward, and let B be the ball $x^2 + y^2 + z^2 \leq 4$. Then by the Divergence Theorem,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot n\, dS &= \iiint_B \nabla \cdot \mathbf{F}\, dV \\
&= \iiint_B \left(3z^2 + 3y^2 + 3x^2 \right)\, dV \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^2 3\,\rho^2\,\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \\
&= \frac{3}{5}(2^5) \int_0^{2\pi} \int_0^{\pi} \sin\varphi\, d\varphi\, d\theta \\
&= \frac{192}{5} \int_0^{2\pi} d\theta \\
&= \frac{384\pi}{5}.
\end{aligned}$$

End of Exam.