Tufts University

Math 42 — Department of Mathematics — May 6, 2013

Final Examination — 8:30 – 10:30 a.m.

Instructions: Please do all 10 problems below. Except for Problem 1, you must show your work and justify your answers in order for partial or full credit to be awarded. No books, notes or calculators are allowed during this exam. *You are required to sign each examination blue book that you are handing in. With your signature, you are pledging that you have neither given nor received any help pertaining to this exam. If you are found in violation of this policy, you will be refered to the Dean of Students and automatically receive an* **F** *for the course.*

1. (10 points) **True or False – no partial credit.** On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) The line $x = 2 - t,\ y = 1 + 3t,\ z = 1 + t$ is perpendicular to the plane $2x - 6y - 2z = 7$.

   (b) The vector $\mathbf{v} = 4\,\mathbf{i} - 2\,\mathbf{j} - \mathbf{k}$ is normal to the surface $x^2 - y^2 z = 3$ at the point $(2, 1, 1)$.

   (c) The line integral $\displaystyle\int_C x\cos y\,dx - x^2 \sin y\,dy$ in $\mathbb{R}^2$ is independent of path.

   (d) $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} xy\,dy\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r^3 \sin\theta\,\cos\theta\,dr\,d\theta.$

   (e) If $\mathbf{F}(x, y, z) = (x^3y^2 + z^3)\,\mathbf{i} - x^2y^3\,\mathbf{j} + (x + y)\,\mathbf{k}$, then the surface integral
   $$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$
   over any closed oriented surface S is equal to 0.

2. (10 points) The temperature T at the point (x, y, z) in $\mathbb{R}^3$ is given by $T(x, y, z) = e^x\,y^2\,z$.

   (a) Find the gradient ∇T of the temperature at the point $(0, -1, 2)$.

   (b) At the point $(0, -1, 2)$, find the direction towards which the temperature increases most rapidly. Express the direction as a unit vector.

   (c) A gnat is flying along a path such that at the instant it is at the point $(0, -1, 2)$, its velocity vector is $\mathbf{v} = 3\,\mathbf{i} + 2\mathbf{j} + 2\,\mathbf{k}$. What is the rate of change of temperature along its path at this instant?

3. (10 points) Let $f(x, y) = xy - x^2y - xy^2$.

   (a) The point $(\frac{1}{3}, \frac{1}{3})$ is a critical point of $f(x, y)$. Find the other *three* critical points of $f(x, y)$.

   (b) Determine whether the critical point $(\frac{1}{3}, \frac{1}{3})$ gives a local maximum, a local minimum, or a saddle point for $f(x, y)$.

The exam continues on the other side of this sheet.

4. (10 points) The method of Lagrange multipliers is to be used to maximize the function $f(x, y) = 4x + 2y$ on the ellipse $\dfrac{x^2}{2} + y^2 = 1$.

   (a) Write down the system of three equations in x, y, and λ which you will need to find the point (x, y) on the ellipse which maximizes the function f.

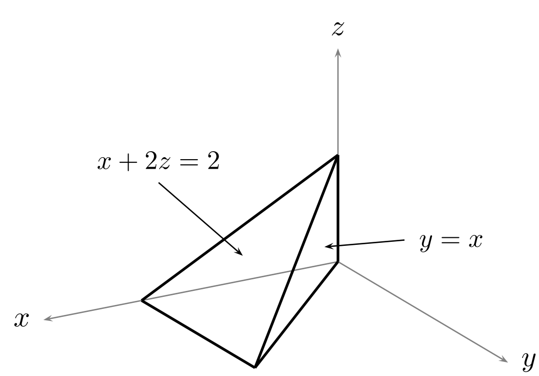
   (b) Using your answer to Part (a), find the point (x, y) on the ellipse which maximizes the function f.

5. (10 points)

   (a) The figure below shows the region of integration for the integral

$$\int_0^2 \int_0^x \int_0^{1-\frac{1}{2}x} f(x, y, z)\, dz\, dy\, dx.$$

   Rewrite this integral as an equivalent iterated integral in the order $dx\, dz\, dy$.



   (b) Express the triple integral

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} xyz^2\, dz\, dy\, dx$$

   as an iterated triple integral in spherical coordinates. DO NOT EVALUATE.

6. (10 points)

   (a) Let $\mathbf{F}(x, y, z) = e^x yz\,\mathbf{i} + (e^x z + 2yz^2)\,\mathbf{j} + (e^x y + 2y^2 z + 3z^2)\,\mathbf{k}$. Find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral

$$\int_C e^x yz\,dx + (e^x z + 2yz^2)\,dy + (e^x y + 2y^2 z + 3z^2)\,dz,$$

where C is the line segment from $(0,0,0)$ to $(2,1,1)$.

The exam continues on the next page.

7. (10 points) Use Green's Theorem to evaluate the line integral

$$\oint_C (x^6 + xy)\,dx + x\,dy$$

where C is the closed curve consisting of the semicircle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$, followed by the line segment from $(-2, 0)$ to $(2, 0)$.

8. (10 points) Consider the surface S (called the *helicoid*) given by the parametric equations

$$x = u\cos v \qquad y = u\sin v \qquad z = v \qquad (0 \le u \le 2,\ 0 \le v \le 2\pi)$$

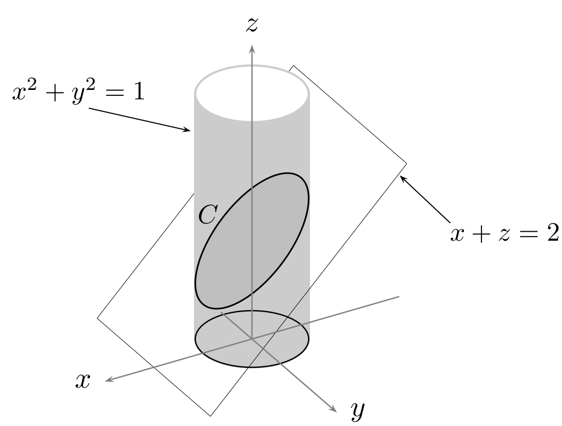
Express the surface area of S as an iterated double integral in u and v. DO NOT EVALUATE.

9. (10 points) Consider the vector field

$$\mathbf{F}(x, y, z) = xz^2\,\mathbf{i} + xy^2\,\mathbf{j} + x^2y\,\mathbf{k}.$$

(a) Find the curl $\nabla \times \mathbf{F}$ of the vector field $\mathbf{F}(x, y, z)$.

(b) Use Stokes' theorem to compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve of intersection of the plane $x + z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise when viewed from above.



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10. (10 points) Use the divergence theorem to compute the *outward* flux of the vector field

$$\mathbf{F}(x, y, z) = 3xz^2\,\mathbf{i} + y^3\,\mathbf{j} + 3x^2z\,\mathbf{k}$$

across the sphere $x^2 + y^2 + z^2 = 4$.

End of Exam.

George Green
(1793-1841)

Sir George Gabriel Stokes
(1819-1903)

Carl Friedrich Gauss
(1777-1855)