1. (10 points) **True or False – no partial credit.** On the first page of your blue book, answer the following questions as **True** or **False**.

(a) If $D$ is the disk in the plane given by $x^2 + y^2 \leq r^2$, then
\[
\int \int_D \sqrt{r^2 - x^2 - y^2} \, dx \, dy = \frac{1}{2} \cdot \frac{4}{3} \pi r^3.
\]
**True:** The double integral represents the volume of the solid above the $(x, y)$-plane and below the hemisphere $z = \sqrt{r^2 - x^2 - y^2}$, and so equals half the volume of a sphere of radius $r$. Thus the integral equals $\frac{1}{2} \cdot \frac{4}{3} \pi r^3$.

(b) \[
\int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{x^2+y^2} \, dy \, dx = \int_{0}^{2\pi} \int_{0}^{1} r^2 \, dr \, d\theta.
\]
**False:** The correct equation is \[
\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{x^2+y^2} \, dy \, dx = \int_{0}^{2\pi} \int_{0}^{1} e^{r^2} \, r \, dr \, d\theta.
\]

(c) If the point $P$ has spherical coordinates $(\rho, \varphi, \theta) = (4, \pi/4, \pi/3)$, then its Cartesian coordinates are $(x, y, z) = (\sqrt{2}, \sqrt{6}, 2\sqrt{2})$.
**True:** Using the conversion equations, we have
\[
x = \rho \sin \varphi \cos \theta = 4 \sin(\pi/4) \cos(\pi/3) = 4(\sqrt{2}/2)(1/2) = \sqrt{2}
\]
\[
y = \rho \sin \varphi \sin \theta = 4 \sin(\pi/4) \sin(\pi/3) = 4(\sqrt{2}/2)(\sqrt{3}/2) = \sqrt{6}
\]
\[
z = \rho \cos \varphi = 4 \cos(\pi/4) = 4(\sqrt{2}/2) = 2\sqrt{2}.
\]

(d) If $S$ is a surface whose equation in spherical coordinates is $\rho \cos \varphi = 3$, then $S$ is a plane.
**True:** Since $z = \rho \cos \varphi$, $S$ has Cartesian equation $z = 3$.

(e) \[
\int_{0}^{2} \int_{0}^{x} f(x, y) \, dy \, dx = \int_{0}^{2} \int_{0}^{y} f(x, y) \, dx \, dy.
\]
**False:** The left hand side is an iterated integral over the triangular region $0 \leq y \leq x$, $0 \leq x \leq 2$, shown below:
Thus, changing the order of integration of the integral on the left would give \( \int_0^2 \int_y^2 f(x, y) \, dx \, dy \).

2. (12 points) A box with faces parallel to the coordinate planes lies in the first octant inside the ellipsoid \( \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1 \). (See the figure below.)

The volume of the largest such box is to be found using Lagrange multipliers.

(a) Write down the system of four equations in \( x, y, z \), and \( \lambda \) which you will need to solve for the vertex \( (x, y, z) \) of the largest box.

**Solution:** If \( (x, y, z) \) is the vertex of the box which lies in the ellipsoid, then we want to maximize the volume function \( f(x, y, z) = xyz \) subject to the constraint \( g(x, y, z) = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1 \). At the point where \( f \) is maximized, we must have \( \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \), which gives:

\[
\begin{align*}
  f_x &= \lambda g_x \implies yz = \lambda \cdot \frac{2x}{25} \\
  f_y &= \lambda g_y \implies xz = \lambda \cdot \frac{2y}{16} \\
  f_z &= \lambda g_z \implies xy = \lambda \cdot \frac{2z}{9} \\
  g(x, y, z) &= 1 \implies \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1.
\end{align*}
\]

(b) Solve the system you obtained in Part (a) for \( x, y, \) and \( z \).

**Solution:** Multiplying the first three equations above by \( x, y, \) and \( z \), respectively, we obtain:

\[
xyz = \lambda \cdot \frac{2x^2}{25} = \lambda \cdot \frac{2y^2}{16} = \lambda \cdot \frac{2z^2}{9}.
\]

Note that \( \lambda \) cannot be zero: otherwise from the first three equations in Part (a), at least one of \( x, y, \) or \( z \) equals zero, yielding zero volume. Thus we can cancel \( \lambda \) from the last
equalities above, which gives us:

\[
x^2 = \frac{y^2}{25} = \frac{z^2}{16} = \frac{z^2}{9}.
\]

Since all three terms on the left hand side of the constraint equation \(g(x, y, z) = 1\) are equal, we obtain

\[
3 \cdot \frac{x^2}{25} = 1 \implies x^2 = \frac{25}{3} \implies x = \frac{5}{\sqrt{3}}
\]

\[
3 \cdot \frac{y^2}{16} = 1 \implies y^2 = \frac{16}{3} \implies y = \frac{4}{\sqrt{3}}
\]

\[
3 \cdot \frac{z^2}{9} = 1 \implies z^2 = \frac{9}{3} \implies z = \sqrt{3}.
\]

Since there is only one critical point, this must yield the maximum volume. This maximum volume equals \(\frac{5}{\sqrt{3}} \cdot \frac{4}{\sqrt{3}} \cdot \sqrt{3} = \frac{20\sqrt{3}}{3}\).

3. (10 points) Consider the double integral:

\[
\int_0^2 \int_{x^2}^4 x e^{y^2} \, dy \, dx.
\]

(a) Sketch the region of integration and label the boundary curves.

**Solution:**

(b) Switch the order of integration and evaluate the double integral.
Solution:

\[
\int_0^4 \int_0^{\sqrt{y}} x e^{y^2} \, dx \, dy = \int_0^4 \left[ \frac{x^2}{2} e^{y^2} \right]_0^{\sqrt{y}} \, dy \\
= \int_0^4 \frac{y}{2} e^{y^2} \, dy \\
= \frac{1}{4} e^{y^2} \bigg|_0^4 \\
= e^{16} - 1.
\]

4. (10 points) Let \( R \) be the region in the plane outside the unit circle \( r = 1 \) and inside the circle \( r = 2 \cos \theta \). (See the figure below.)

(a) The two circles intersect at the points \( P \) and \( Q \). Find the polar coordinates of \( P \) and \( Q \).

**Solution:** We solve the equations \( r = 1 \) and \( r = 2 \cos \theta \) simultaneously: equating the right hand sides gives us \( 1 = 2 \cos \theta \implies \cos \theta = 1/2 \implies \theta = \pi/3 \) or \( \theta = -\pi/3 \). The point \( P \) therefore has polar coordinates \((1, \pi/3)\) and \( Q \) has polar coordinates \((1, -\pi/3)\).

(b) Express the double integral

\[
\int\int_{R} \frac{1}{1 + x^2 + y^2} \, dx \, dy
\]

as an iterated double integral in polar coordinates. **DO NOT EVALUATE.**

**Solution:**

\[
\int\int_{R} \frac{1}{1 + x^2 + y^2} \, dx \, dy = \int_{-\pi/3}^{\pi/3} \int_{1}^{2 \cos \theta} \frac{1}{1 + r^2} \, r \, dr \, d\theta.
\]

5. (12 points) Let \( E \) be the solid in the first octant bounded by the parabolic cylinder \( z = 1 - x^2 \) and by the plane \( y = 1 - x \). (See the figure below.)
Express the triple integral $\iiint_E f(x, y, z) \, dV$ as an iterated triple integral

(a) in the order $dz \, dy \, dx$

Solution: \[
\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x, y, z) \, dz \, dy \, dx.
\]

(b) in the order $dy \, dx \, dz$

Solution: \[
\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x, y, z) \, dy \, dx \, dz.
\]

6. (12 points) Express the following iterated triple integral

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} (x^2 + y^2 + z^2)^{\frac{3}{2}} \, dz \, dy \, dx$$

in spherical coordinates. **DO NOT EVALUATE.**

**Solution:** The triple integral is taken over the solid $E$ below the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$. (See the picture below.)
In spherical coordinates, these surfaces have equations given by $\rho = 2$ and $\varphi = \pi/4$, respectively. Thus, we may write the above triple integral in spherical coordinates as follows:

$$
\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} (x^2 + y^2 + z^2)^{\frac{3}{2}} \, dz \, dy \, dx = \iiint_E (x^2 + y^2 + z^2)^{\frac{3}{2}} \, dV
$$

$$
= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\rho} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta
$$

$$
= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\rho} \rho^5 \sin \varphi \, d\rho \, d\varphi \, d\theta
$$

7. (12 points) Let $E$ be the solid outside the cylinder $x^2 + y^2 = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

(a) Express the volume of $E$ as an iterated triple integral in cylindrical coordinates.

**Solution:** The top half of the solid $E$ (i.e., the part of $E$ above the $(x,y)$-plane) is shown in the picture below.

![Diagram of the solid E](image)

and a sketch of the projection of the solid region onto the $xy$-plane is

![Projection of the solid region onto the xy-plane](image)
The volume of $E$ is therefore given by
\[
\iiint_E dV = \int_0^{2\pi} \int_1^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta
\]

(b) Evaluate the integral you obtained in Part (a).

**Solution:** The right hand integral in Part (a) equals
\[
\int_0^{2\pi} \int_1^2 [rz/\sqrt{4-r^2}] \, dr \, d\theta = \int_0^{2\pi} \int_1^2 2r \sqrt{4-r^2} \, dr \, d\theta
\]
\[
= \int_0^{2\pi} \left[ -\frac{2}{3} (4-r^2)^{3/2} \right]_1^2 \, d\theta
\]
\[
= \frac{2}{3} \int_0^{2\pi} 2 \, d\theta
\]
\[
= \frac{2}{3} \cdot 3 \cdot 2\pi
\]
\[
= 4\pi \sqrt{3}.
\]

8. (12 points) Evaluate the line integral
\[
\int_C x \, ds,
\]
where $C$ is the arc of the helix $x = 3 \cos t$, $y = 3 \sin t$, $z = 4t$ from $(3, 0, 0)$ to $(0, 3, 2\pi)$.

**Solution:** The intial point $(3, 0, 0)$ of the helix occurs when $t = 0$ and the terminal point $(0, 3, 2\pi)$ occurs when $t = \pi/2$. The helix itself is given by the vector function $\mathbf{r}(t) = 3 \cos t \, \mathbf{i} + 3 \sin t \, \mathbf{j} + 4t \, \mathbf{k}$. Hence
\[
\mathbf{r}'(t) = -3 \sin t \, \mathbf{i} + 3 \cos t \, \mathbf{j} + 4 \, \mathbf{k}
\]
and
\[
|\mathbf{r}'(t)| = \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + 4^2} = 5.
\]

It follows that
\[
\int_C x \, ds = \int_0^{\pi/2} 3 \cos t \, |\mathbf{r}'(t)| \, dt
\]
\[
= \int_0^{\pi/2} 3 \cos t \cdot 5 \, dt
\]
\[
= 15 \int_0^{\pi/2} \cos t \, dt
\]
\[
= 15.
\]

9. (10 points) Evaluate the line integral
\[
\int_C \mathbf{F} \cdot \mathbf{dr},
\]

[The rest of the text is not displayed here as it is not relevant to the given question.]
where \( \mathbf{F} = (y + z) \mathbf{i} - 2x \mathbf{j} + 3z \mathbf{k} \), and \( C \) is the curve parametrized by \( \mathbf{r}(t) = (t^2, t, t^3) \) for \( 0 \leq t \leq 1 \).

**Solution:** The curve has parametric equations \( x = t^2, y = t, z = t^3 \), for \( 0 \leq t \leq 1 \). Hence

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (y + z) \, dx - 2x \, dy + 3z \, dz
\]

\[
= \int_0^1 [(t + t^3) \, (2t \, dt) - 2t^2 \, (dt) + 3t^3 \, (3t^2 \, dt)]
\]

\[
= \int_0^1 [9t^5 + 2t^4] \, dt
\]

\[
= \frac{9}{6} + \frac{2}{5}
\]

\[
= \frac{19}{10}
\]

*End of Exam.*