Tufts University

Math 42 — Department of Mathematics

Exam 2 Solutions

1. (10 points) **True or False – no partial credit.** On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) If D is the disk in the plane given by $x^2 + y^2 \leq r^2$, then

   $$\iint_D \sqrt{r^2 - x^2 - y^2}\, dx\, dy = \frac{1}{2} \cdot \frac{4}{3}\pi r^3.$$

   True: The double integral represents the volume of the solid above the (x, y)-plane and below the hemisphere $z = \sqrt{r^2 - x^2 - y^2}$, and so equals half the volume of a sphere of radius r. Thus the integral equals $\frac{1}{2} \cdot \frac{4}{3}\pi r^3$.

   (b) $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx = \int_0^{2\pi} \int_0^1 e^{r^2}\, dr\, d\theta.$

   False: The correct equation is $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx = \int_0^{2\pi} \int_0^1 e^{r^2}\, r\, dr\, d\theta$

   (c) If the point P has spherical coordinates $(\rho, \varphi, \theta) = (4, \pi/4, \pi/3)$, then its Cartesian coordinates are $(x, y, z) = (\sqrt{2}, \sqrt{6}, 2\sqrt{2})$.

   True: Using the conversion equations, we have

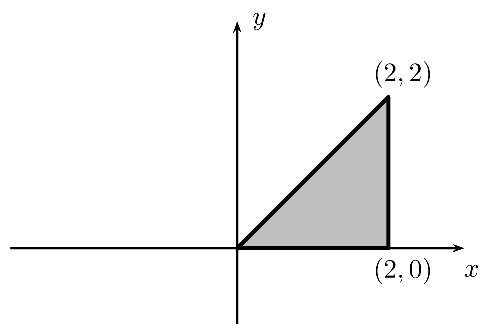
   $$\begin{aligned} x &= \rho \sin\varphi \cos\theta = 4 \sin(\pi/4) \cos(\pi/3) = 4(\sqrt{2}/2)(1/2) = \sqrt{2} \\ y &= \rho \sin\varphi \sin\theta = 4 \sin(\pi/4) \sin(\pi/3) = 4(\sqrt{2}/2)(\sqrt{3}/2) = \sqrt{6} \\ z &= \rho \cos\varphi = 4\cos(\pi/4) = 4(\sqrt{2}/2) = 2\sqrt{2}. \end{aligned}$$

   (d) If S is a surface whose equation in spherical coordinates is $\rho \cos\varphi = 3$, then S is a plane.

   True: Since $z = \rho \cos\varphi$, S has Cartesian equation $z = 3$.

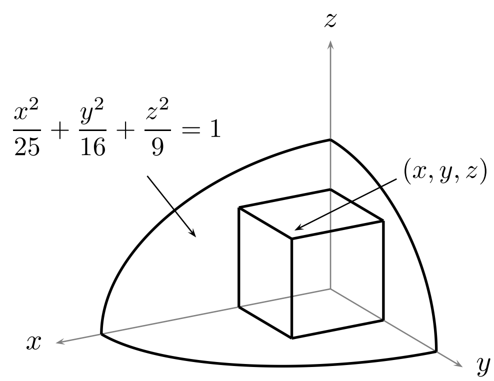
   (e) $\int_0^2 \int_0^x f(x, y)\, dy\, dx = \int_0^2 \int_0^y f(x, y)\, dx\, dy.$

   False: The left hand side is an iterated integral over the triangular region $0 \leq y \leq x$, $0 \leq x \leq 2$, shown below:



Thus, changing the order of integration of the integral on the left would give $\int_0^2 \int_y^2 f(x,y)\,dx\,dy$.

2. (12 points) A box with faces parallel to the coordinate planes lies in the first octant inside the ellipsoid $\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1$. (See the figure below.)
The volume of the largest such box is to be found using Lagrange multipliers.



(a) Write down the system of four equations in x, y, z, and λ which you will need to solve for the vertex (x, y, z) of the largest box.

Solution: If (x, y, z) is the vertex of the box which lies in the ellipsoid, then we want to maximize the volume function $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1$. At the point where f is maximized, we must have $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, which gives:

$$\begin{aligned} f_x = \lambda\, g_x &\implies yz = \lambda \cdot \frac{2x}{25} \\ f_y = \lambda\, g_y &\implies xz = \lambda \cdot \frac{2y}{16} \\ f_x = \lambda\, g_x &\implies xy = \lambda \cdot \frac{2z}{9} \\ g(x, y, z) = 1 &\implies \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1. \end{aligned}$$

(b) Solve the system you obtained in Part (a) for x, y, and z.

Solution: Multiplying the first three equations above by x, y, and z, respectively, we obtain

$$xyz = \lambda \cdot \frac{2x^2}{25} = \lambda \cdot \frac{2y^2}{16} = \lambda \cdot \frac{2z^2}{9}.$$

Note that λ cannot be zero: otherwise from the first three equations in Part (a), at least one of x, y, or z equals zero, yielding zero volume. Thus we can cancel λ from the last equalities above, which gives us:

$$\frac{x^2}{25} = \frac{y^2}{16} = \frac{z^2}{9}.$$

Since all three terms on the left hand side of the constraint equation $g(x, y, z) = 1$ are equal, we obtain

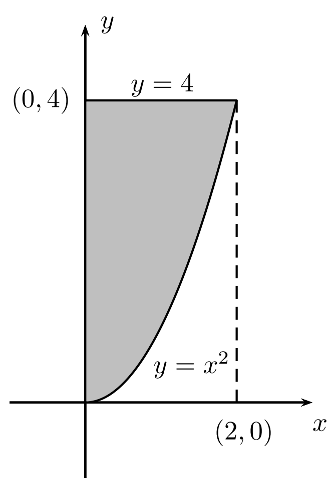
$$3 \cdot \frac{x^2}{25} = 1 \implies x^2 = \frac{25}{3} \implies x = \frac{5}{\sqrt{3}}$$
$$3 \cdot \frac{y^2}{16} = 1 \implies y^2 = \frac{16}{3} \implies y = \frac{4}{\sqrt{3}}$$
$$3 \cdot \frac{z^2}{9} = 1 \implies z^2 = \frac{9}{3} \implies z = \sqrt{3}.$$

Since there is only one critical point, this must yield the maximum volume. This maximum volume equals $\frac{5}{\sqrt{3}} \cdot \frac{4}{\sqrt{3}} \cdot \sqrt{3} = \frac{20\sqrt{3}}{3}$.

3. (10 points) Consider the double integral:

$$\int_0^2 \int_{x^2}^4 x\, e^{y^2}\, dy\, dx.$$

(a) Sketch the region of integration and label the boundary curves.
Solution:

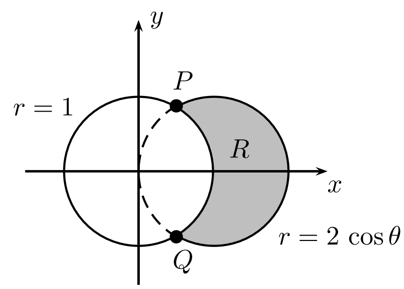


(b) Switch the order of integration and evaluate the double integral.

Solution:

$$\int_0^4 \int_0^{\sqrt{y}} x\, e^{y^2}\, dx\, dy = \int_0^4 \left[\frac{x^2}{2}\, e^{y^2} \right]_0^{\sqrt{y}} dy$$
$$= \int_0^4 \frac{y}{2}\, e^{y^2}\, dy$$
$$= \frac{1}{4}\, e^{y^2} \Big|_0^4$$
$$= \frac{e^{16}-1}{4}.$$

4. (10 points) Let R be the region in the plane outside the unit circle $r = 1$ and inside the circle $r = 2\cos\theta$. (See the figure below.)



(a) The two circles intersect at the points P and Q. Find the polar coordinates of P and Q.
Solution: We solve the equations $r = 1$ and $r = 2\cos\theta$ simultaneously: equating the right hand sides gives us $1 = 2\cos\theta \implies \cos\theta = 1/2 \implies \theta = \pi/3$ or $\theta = -\pi/3$. The point P therefore has polar coordinates $(1, \pi/3)$ and Q has polar coordinates $(1, -\pi/3)$.

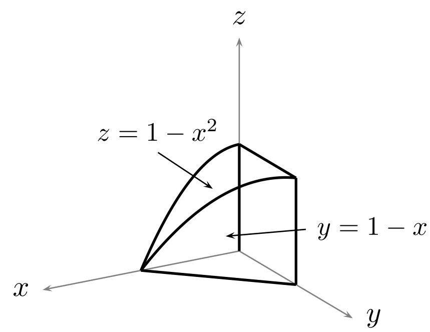
(b) Express the double integral

$$\iint_R \frac{1}{1+x^2+y^2}\, dx\, dy$$

as an iterated double integral in polar coordinates. **DO NOT EVALUATE.**
Solution:

$$\iint_R \frac{1}{1+x^2+y^2}\, dx\, dy = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{2\cos\theta} \frac{1}{1+r^2}\, r\, dr\, d\theta.$$

5. (12 points) Let E be the solid in the first octant bounded by the parabolic cylinder $z = 1 - x^2$ and by the plane $y = 1 - x$. (See the figure below.)



Express the triple integral $\iiint_E f(x,y,z)\,dV$ as an iterated triple integral

(a) in the order $dz\,dy\,dx$

Solution: $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx.$

(b) in the order $dy\,dx\,dz$.

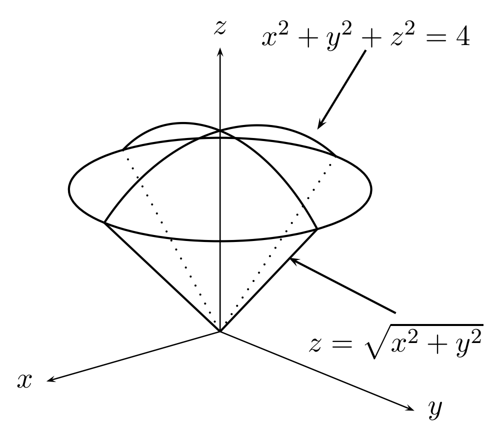
Solution: $\displaystyle\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz.$

6. (12 points) Express the following iterated triple integral

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} \left(x^2+y^2+z^2\right)^{\frac{3}{2}}\,dz\,dy\,dx$$

in spherical coordinates. **DO NOT EVALUATE.**

Solution: The triple integral is taken over the solid E below the sphere $x^2+y^2+z^2=4$ and above the cone $z=\sqrt{x^2+y^2}$. (See the picture below.)

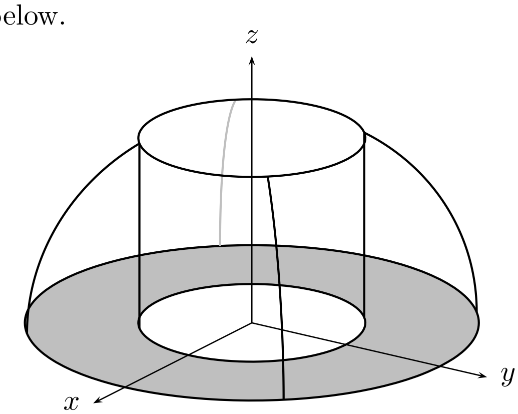


In spherical coordinates, these surfaces have equations given by $\rho = 2$ and $\varphi = \pi/4$, respectively. Thus, we may write the above triple integral in spherical coordinates as follows:

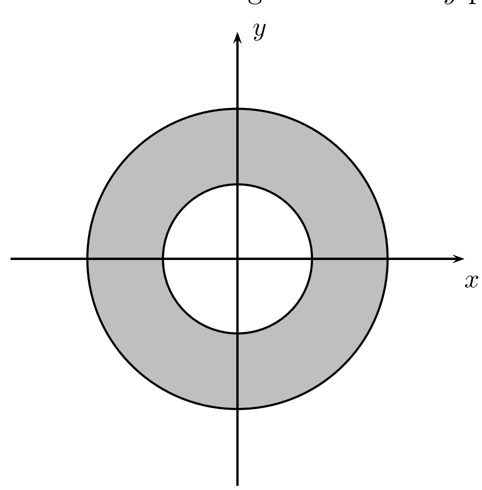
$$\begin{aligned}
\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} \left(x^2+y^2+z^2\right)^{\frac{3}{2}} \, dz\, dy\, dx &= \iiint_E (x^2+y^2+z^2)^{\frac{3}{2}} \, dV \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^2 (\rho^2)^{\frac{3}{2}} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^2 \rho^5 \sin\varphi \, d\rho\, d\varphi\, d\theta
\end{aligned}$$

7. (12 points) Let E be the solid outside the cylinder $x^2 + y^2 = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

   (a) Express the volume of E as an iterated triple integral in cylindrical coordinates.
   Solution: The top half of the solid E (i.e., the part of E above the (x, y)-plane) is shown in the picture below.



and a sketch of the projection of the solid region onto the xy-plane is



The volume of E is therefore given by

$$\iiint_E dV = \int_0^{2\pi} \int_1^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$$

(b) Evaluate the integral you obtained in Part (a).

Solution: The right hand integral in Part (a) equals

$$\begin{aligned}
\int_0^{2\pi} \int_1^2 [rz]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}}\, dr\, d\theta &= \int_0^{2\pi} \int_1^2 2r\sqrt{4-r^2}\, dr\, d\theta \\
&= \int_0^{2\pi} \left[-\frac{2}{3}\left(4-r^2\right)^{\frac{3}{2}} \right]_1^2 d\theta \\
&= \frac{2}{3} 3^{\frac{3}{2}} \int_0^{2\pi} d\theta \\
&= \frac{2}{3} 3^{\frac{3}{2}} \cdot 2\pi \\
&= 4\pi\sqrt{3}.
\end{aligned}$$

8. (12 points) Evaluate the line integral

$$\int_C x\, ds,$$

where C is the arc of the helix $x = 3\cos t$, $y = 3\sin t$, $z = 4t$ from $(3, 0, 0)$ to $(0, 3, 2\pi)$.

Solution: The intial point $(3, 0, 0)$ of the helix occurs when $t = 0$ and the terminal point $(0, 3, 2\pi)$ occurs when $t = \pi/2$. The helix itself is given by the vector function $\mathbf{r}(t) = 3\cos t\, \mathbf{i} + 3\sin t\, \mathbf{j} + 4t\, \mathbf{k}$. Hence

$$\mathbf{r}'(t) = -3\sin t\, \mathbf{i} + 3\cos t\, \mathbf{j} + 4\, \mathbf{k}$$

and

$$|\mathbf{r}'(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = 5.$$

It follows that

$$\begin{aligned}
\int_C x\, ds &= \int_0^{\frac{\pi}{2}} 3\cos t\, |\mathbf{r}'(t)|\, dt \\
&= \int_0^{\frac{\pi}{2}} 3\cos t \cdot 5\, dt \\
&= 15 \int_0^{\frac{\pi}{2}} \cos t\, dt \\
&= 15.
\end{aligned}$$

9. (10 points) Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = (y+z)\,\mathbf{i} - 2x\,\mathbf{j} + 3z\,\mathbf{k}$, and C is the curve parametrized by $\mathbf{r}(t) = \langle t^2, t, t^3 \rangle$ for $0 \le t \le 1$.

Solution: The curve has parametric equations $x = t^2$, $y = t$, $z = t^3$, for $0 \le t \le 1$. Hence

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (y+z)\,dx - 2x\,dy + 3z\,dz \\
&= \int_0^1 \left[(t+t^3)\,(2t\,dt) - 2t^2\,(dt) + 3t^3\,(3t^2\,dt)\right] \\
&= \int_0^1 \left[9t^5 + 2t^4\right]\,dt \\
&= \frac{9}{6} + \frac{2}{5} \\
&= \frac{19}{10}.
\end{aligned}$$

End of Exam.