Tufts University

Math 42 | Department of Mathematics | Spring 2013

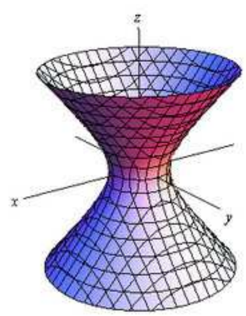
Exam 1 Solutions

1. (10 points) **True or False – no partial credit.** On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) The cross product of two unit vectors is a unit vector.
   Solution: False: for instance, $\mathbf{i} \times \mathbf{i} = \mathbf{0}$.

   (b) The line $x = 3-2t,\ y = 4+4t,\ z = -1-2t$ is perpendicular to the plane $x-2y+z = 7$.
   Solution: True: The line is parallel to the vector $\mathbf{v} = -2\,\mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k}$, the plane has normal vector $\mathbf{n} = \mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$, and since $\mathbf{v} = -2\,\mathbf{n}$, the two vectors are parallel.

   (c) The surface below is the graph of the equation $x^2 + y^2 - 3z^2 = -1$.



   Solution: False: the graph of the equation is a hyperboloid of two sheets. This can also be seen by noting that the graph of the equation has *no* (x, y)-trace.

   (d) The curve $\mathbf{r}(t) = 3\,\sin t\,\mathbf{i} - 3\,\cos t\,\mathbf{j} + 4t\,\mathbf{k}$ uses arc length as a parameter.
   Solution: False: the derivative $\mathbf{r}'(t) = 3\,\cos t\,\mathbf{i} + 3\,\sin t\,\mathbf{j} + 4\,\mathbf{k}$ has length $|\mathbf{r}'(t)| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = 5 \neq 1$.

   (e) If $\nabla f(a, b) = \mathbf{0}$, then the tangent plane to the graph of $z = f(x, y)$ above the point (a, b) is parallel to the (x, y)-plane.
   Solution: True: the tangent plane above (a, b) has normal vector $\langle -f_x(a, b), -f_y(a, b), 1\rangle = \langle 0, 0, 1\rangle$.

2. (12 points) Consider the triangle with vertices $P(3, 1, 0)$, $Q(4, 3, 1)$, and $R(5, 2, 0)$.

   (a) Find the area of the triangle ΔPQR.
   Solution: The area of the triangle is $\frac{1}{2}|\vec{PQ} \times \vec{PR}|$. We calculate $\vec{PQ} \times \vec{PR}$:

$$
\begin{aligned}
\vec{PQ} \times \vec{PR} &= (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \times (2\mathbf{1} + \mathbf{j}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} \\
&= -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}
\end{aligned}
$$

It follows that

$$\begin{aligned}\text{area}(\Delta PQR) &= \frac{1}{2}\sqrt{(-1)^2+2^2+(-3)^2}\\ &= \frac{\sqrt{14}}{2}\end{aligned}$$

(b) Find an equation of the plane containing the triangle ΔPQR.

Solution: A normal vector to the plane containing the points P, Q, and R is $\mathbf{n} = \vec{PQ} \times \vec{PR} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Using the point $P(3,1,0)$ as a point through which the plane passes, we see that the plane has equation

$$-(x-3) + 2(y-1) - 3(z-0) = 0,$$

which simplifies to

$$x - 2y + 3z = 1$$

3. (6 points) Let $\mathbf{u} = 5\,\mathbf{i} + 3\,\mathbf{j} + 8\,\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$.

(a) Find the scalar projection $\text{scal}_{\mathbf{v}}\mathbf{u}$.

Solution:

$$\begin{aligned}\text{scal}_{\mathbf{v}}\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\\ &= \frac{18}{\sqrt{6}}\\ &= 3\sqrt{6}\end{aligned}$$

(b) Find the scalar projection $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Solution:

$$\begin{aligned}\text{proj}_{\mathbf{v}}\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}\\ &= \frac{18}{6}\,(\mathbf{i} - \mathbf{j} + 2\,\mathbf{k})\\ &= 3\,\mathbf{i} - 3\,\mathbf{j} + 6\,\mathbf{k}.\end{aligned}$$

4. (15 points) A disoriented mosquito is flying along a path such that its position at time t is $\mathbf{r}(t) = 3\,\cos t\,\mathbf{i} + 4\,\cos t\,\mathbf{j} + 5\,\sin t\,\mathbf{k}$.

(a) Determine the mosquito's velocity and acceleration when $t = \frac{\pi}{2}$.

Solution: The velocity and acceleration are given by

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = -3\,\sin t\,\mathbf{i} - 4\,\sin t\,\mathbf{j} + 5\,\cos t\,\mathbf{k}\\ \mathbf{a}(t) &= \mathbf{v}'(t) = -3\,\cos t\,\mathbf{i} - 4\,\cos t\,\mathbf{j} - 5\,\sin t\,\mathbf{k} \quad (= -\mathbf{r}(t)!)\end{aligned}$$

When $t = \pi/2$, the velocity is given by:

$$\mathbf{v}(\pi/2) = -3\,\mathbf{i} - 4\,\mathbf{j}$$

The acceleration at $t = \pi/2$ equals

$$\mathbf{a}(\pi/2) = -5\,\mathbf{k}.$$

(b) Find parametric equations for the tangent line to the curve above when $t = \frac{\pi}{3}$.
The derivative $\mathbf{r}'(\pi/2) = \mathbf{v}(\pi/2)$ is parallel to the tangent line. Thus our tangent line has direction numbers -3, -4, 0. When $t = \pi/2$, we also have

$$\mathbf{r}(\pi/2) = 5\,\mathbf{k},$$

so the mosquito is at the point $(0, 0, 5)$. Therefore the tangent line has parametric equations

$$x = -3t, \quad y = -4t, \quad z = 5.$$

(c) Calculate the distance the mosquito has traveled from $t = 0$ to $t = 2$.
Solution: It turns out that the mosquito travels at constant speed, since from Part (a), we obtain

$$\begin{aligned} |\mathbf{v}(t)| &= \left((-3\sin t)^2 + (-4\sin t)^2 + (5\cos t)^2\right)^{\frac{1}{2}} \\ &= \left((3^2 + 4^2)\,\sin^2 t + 5^2\,\cos^2 t\right)^{\frac{1}{2}} \\ &= 5. \end{aligned}$$

Hence the distance traveled by the mosquito from $t = 0$ to $t = 2$ is the arc length

$$\int_0^2 |\mathbf{r}'(t)|\,dt = \int_0^2 5\,dt = 10.$$

5. (10 points) Find parametric equations of the line of intersection of the two planes $x+y+z = 3$ and $2x - y + 3z = 4$. (You may use the fact that the point $P(1, 1, 1)$ lies in both planes.)

Solution: The two planes have normal vectors $\mathbf{n}_1 = \mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{n}_2 = 2\,\mathbf{i}-\mathbf{j}+3\,\mathbf{k}$, respectively. The line of intersection is orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$, so it must be parallel to

$$\begin{aligned} \mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} \\ &= 4\,\mathbf{i} - \mathbf{j} - 3\,\mathbf{k}. \end{aligned}$$

The line of intersection thus has direction numbers 4, -1, -3. Since it passes through the point $P(1, 1, 1)$, it has parametric equations

$$x = 1 + 4t, \quad y = 1 - t, \quad z = 1 - 3t.$$

6. (8 points) Suppose that $z = \dfrac{x}{y}$, and $x = s\,e^t,\ y = s + t$. Find $\dfrac{\partial z}{\partial t}$ when $s = 2$ and $t = 0$.

Solution:

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} \\ &= \frac{1}{y}\cdot se^t - \frac{x}{y^2}\cdot 1\end{aligned}$$

When $s = 2$ and $t = 0$, we have $x = 2$ and $y = 2$. Hence

$$\begin{aligned}\left.\frac{\partial z}{\partial t}\right|_{s=2,\,t=0} &= \frac{1}{2}\cdot 2 - \frac{2}{2^2} \\ &= \frac{1}{2}.\end{aligned}$$

7. (15 points) Let $f(x, y, z) = x^2 - 2y^2 + z^2$.

(a) Find a unit vector in the direction of which $f(x, y, z)$ is increasing the fastest at the point $(1, -1, 2)$.

Solution: Let us calculate the gradient ∇f:

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k} \\ &= 2x\,\mathbf{i} - 4y\,\mathbf{j} + 2z\,\mathbf{k}\end{aligned}$$

At the point $(1, -1, 2)$ the gradient is

$$\nabla f(1, -1, 2) = 2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}$$

Since the gradient points in the direction of most rapd increase for f, a unit vector in the direction of which f is increasing the fastest at $(1, -1, 2)$ is

$$\mathbf{u} = \frac{\nabla f(1, -1, 2)}{|\nabla f(1, -1, 2)|} = \frac{2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}}{6}.$$

(b) Calculate the directional derivative of $f(x, y, z)$ in the direction of the unit vector $\mathbf{v} = \frac{2}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{1}{3}\mathbf{k}$ at the point $(1, -1, 2)$.

Solution: This directional derivative would be

$$\begin{aligned}D_{\mathbf{v}} f(1, -1, 2) &= \nabla f(1, -1, 2)\cdot\mathbf{v} \\ &= (2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k})\cdot\left(\frac{2}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{1}{3}\mathbf{k}\right) \\ &= \frac{16}{3}.\end{aligned}$$

Note: There was a typo in the exam for this problem. It asked to find the directional derivative at $(1, -2, 2)$ *instead of at* $(1, -1, 2)$.

(c) Find an equation of the tangent plane to the level surface $f(x, y, z) = 3$ at the point $(1, -1, 2)$.

Solution: A normal vector to the tangent plane is $\mathbf{n} = \nabla f(1, -1, 2) = 2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}$. Since the plane passes through the point $(1, -1, 2)$, its equation is

$$2(x - 1) + 4(y + 1) + 4(z - 2) = 0,$$

which simplifies to

$$x + 2y + 2z = 3.$$

8. (12 points) Let $f(x, y) = x^2 - xy + 3y^2$.

(a) Find the linear approximation $L(x, y)$ of $f(x, y)$ at the point $(3, -1)$.

Solution: The partial derivatives of f at $(3, -1)$ are

$$\begin{aligned} f_x(3, -1) &= (2x - y)|_{(3,-1)} = 7 \\ f_y(3, -1) &= (-x + 6y)|_{(3,-1)} = -9. \end{aligned}$$

Since $f(3, -1) = 15$, we obtain

$$\begin{aligned} L(x, y) &= f(3, -1) + f_x(3, -1)(x - 3) + f_y(3, -1)(y + 1) \\ &= 15 + 7(x - 3) - 9(y + 1). \end{aligned}$$

(b) Use $L(x, y)$ to estimate $f(2.96, -0.95)$. Give your answer as a decimal expressed to two places.

Solution:

$$\begin{aligned} f(2.96, -0.95) &\approx L(2.96, -0.95) \\ &= 15 + 7(2.96 - 3) - 9(-0.95 + 1) \\ &= 15 - 0.28 - 0.45 \\ &= 14.27. \end{aligned}$$

9. (12 points)

(a) Find all critical points of the function $f(x, y) = xy(1 - x - y)$.

Solution: Write $f(x, y) = xy - x^2y - xy^2$. Let us calculate the partials of f:

$$\begin{aligned} f_x &= y - 2xy - y^2 = y(1 - 2x - y) \\ f_y &= x - x^2 - 2xy = x(1 - x - 2y). \end{aligned}$$

Setting them equal to zero yields:

$$\begin{aligned} f_x = 0 &\implies y = 0 \text{ or } 2x + y = 1 \\ f_y = 0 &\implies x = 0 \text{ or } x + 2y = 1. \end{aligned}$$

This results in four cases, and four critical points:
Case 1: $y = 0$ and $x = 0$. Critical point: $(0, 0)$.
Case 2: $y = 0$ and $x + 2y = 1$. Critical point: $(1, 0)$.
Case 3: $2x + y = 1$ and $x = 0$. Critical point: $(0, 1)$.
Case 4: $2x + y = 1$ and $x + 2y = 1$. Critical point: $(\frac{1}{3}, \frac{1}{3})$.

(b) The origin $(0, 0)$ is a critical point of the function $f(x, y) = x^2y^3 - x^2 + 2y^2$. Determine whether it corresponds to a local maximum, a local minimum, or a saddle point.

Solution: We are going to apply the Second Derivative Test, and for this we will need to calculate all second order partials of f:

$$\begin{aligned} f_x &= 2xy^3 - 2x \\ f_y &= 3x^2y^2 + 4y \\ f_{xx} &= 2y^3 - 2 \\ f_{yy} &= 6x^2y + 4 \\ f_{xy} &= 6xy^2. \end{aligned}$$

At $(0, 0)$, the second order partials are $f_{xx}(0, 0) = -2$, $f_{yy}(0, 0) = 4$, $f_{xy}(0, 0) = 0$. Hence

$$D = f_{xx}(0, 0)\, f_{yy}(0, 0) - (f_{xy}(0, 0))^2 = -8 < 0,$$

and by the Second Derivative Test, the origin yields a *saddle point* for f.

End of Exam.