Math 13 | **Tufts University**
Department of Mathematics
Final Exam | **May 7, 2012**
8:30 am to 10:30 am

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (10 points) **True or False - No Partial Credit**: On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be three nonzero vectors in $\mathbb{R}^3$. Then, $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$.

   False: Both sides of the equality have a scalar (result of the dot product inside the parenthesis) being dotted with a vector. This is not allowed.

   (b) Let $g(x, y)$ be a continuous function defined on the domain $D = \{(x, y) \mid a \le x \le b,\, c \le y \le d\}$. The integral

   $$\iint_D g(x,y)dA = \left(\int_a^b g(x,y)\,dx\right)\left(\int_c^d g(x,y)\,dy\right).$$

   False: The result would be a function of y multiplied by a function of x and not equal to the integral.

   (c) Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

   True: Let $\mathbf{F} = \langle P, Q, R\rangle$. Then, $\nabla \cdot (\nabla \times \mathbf{F}) = \nabla \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$
   $= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = (R_{yx} - R_{xy}) + (Q_{xz} - Q_{zx}) + (P_{zy} - P_{yz}) = 0.$

   (d) Let $\mathbf{F} = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z\rangle$. Stokes' Theorem says that $\iint\limits_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ for $\mathcal{C}$ being the closed boundary curve of the top half of the unit sphere, S.

   False: Stokes' Theorem does not apply since the components of the vector field do not have continuous partial derivatives on the z-axis (x and y are zero) on the sphere.

   (e) Let $\nabla \cdot \mathbf{F} = 1$ and let S be the surface of the unit cube, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, with outward unit normal. Then, $\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = 1$.

   True: Since, S is a smooth surface that bounds the solid cube we have from the divergence theorem, $\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \int_0^1 \nabla \cdot \mathbf{F}\, dV$. Since, $\nabla \cdot \mathbf{F} = 1$, we have

   $$\int_0^1 \int_0^1 \int_0^1 dx\, dy\, dz = 1$$

2. (10 points) Let

$$\mathbf{r}(t) = (2+t)\ \mathbf{i} + (t^2+3)\ \mathbf{j} + (\frac{2}{3}t^3 - 5)\ \mathbf{k}.$$

(a) Find the parametric equations for the tangent line to the curve $\mathbf{r}(t)$ at $t = 3$.

Solution: The point corresponding to $t = 3$ is given by the vector $\mathbf{r}(3) = 5\mathbf{i} + 12\mathbf{j} + 13\mathbf{k}$, so a point on the tangent line is $(5, 12, 13)$. A vector tangent to the curve at any t is computed from the first derivative of the vector-valued function $\mathbf{r}(t)$. This yields $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$, which evaluated at $t = 3$ gives the vector tangent to the curve $\mathbf{r}'(3) = \langle 1, 6, 18\rangle$.

Hence the parametric equations for the tangent line are

$$\begin{cases} x = 5 + t \\ y = 12 + 6t \\ z = 13 + 18t \end{cases} \quad \text{for } -\infty < t < \infty.$$

(b) Find the arc length of $\mathbf{r}(t)$ from $t = 0$ to $t = 3$.

Solution: From part (a) we have $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$. The magnitude of this vector is

$$|\mathbf{r}'(t)| = \sqrt{1 + (2t)^2 + (2t^2)^2} = \sqrt{1 + 4t^2 + 2t^4} = \sqrt{(2t^2+1)^2} = 2t^2 + 1.$$

The arc length is then computed as the integral of the magnitude of the derivative vector from $t = 0$ to $t = 3$.

$$s = \int_0^3 |\mathbf{r}'(t)|\ dt = \int_0^3 (2t^2 + 1)\ dt = \frac{2}{3}t^3 + t\Big|_0^3 = 18 + 3 = 21.$$

3. (10 points) Find the absolute minimum and maximum for the function $f(x, y) = 2x + 2y^2 + 1$ on the unit disk $D = \{(x, y) \mid x^2 + y^2 \le 1\}$.

Solution: In the interior of D, which is the set of points $\{(x, y) \mid x^2 + y^2 < 1\}$, if (x, y) is a critical point of f, then $\nabla f(x, y) = (0, 0)$. But $\nabla f(x, y) = (2, 4y) \neq (0, 0)$. Hence, there can be no extrema in the interior and they have to occur on the boundary of D.

On the boundary, $\{(x, y) \mid x^2 + y^2 = 1\}$, we use the method of Lagrange Multipliers. Let $g(x, y) = x^2 + y^2$, and solve $\nabla f(x, y) = \lambda \nabla g(x, y))$ and $x^2 + y^2 = 1$ simultaneously. Then, we have

$$\begin{cases} 2 = 2\lambda x \\ 4y = 2\lambda y, \\ x^2 + y^2 = 1 \end{cases}$$

Considering the second equation, we have:

$$y(2 - \lambda) = 0.$$

If $y = 0$, then, from the constraint equation,

$$x^2 = 1 \Rightarrow x = \pm 1.$$

Hence, we have $(x, y) = (1, 0)$ or $(x, y) = (-1, 0)$ as candidates.

If $y \neq 0$, then $\lambda = 2$ and from the first equation,

$$2 = 4x \Rightarrow x = \frac{1}{2}.$$

Hence, from the constraint,

$$\frac{1}{4} + y^2 = 1 \Rightarrow y = \pm\frac{\sqrt{3}}{2}.$$

Thus, $(x, y) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ or $(x, y) = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$ are also candidates for the extrema points.

Now, $f(1, 0) = 3$, $f(-1, 0) = -1$, and $f(\frac{1}{2}, \frac{\sqrt{3}}{2}) = f(\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{7}{2}$. Therefore, f has absolute maxima at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ with a value of $\frac{7}{2}$ and the absolute minimum at $(-1, 0)$ with a value of -1.

4. (10 points) Let $f(x, y, z) = x^2 + y^2 + z^2$, and consider the integral

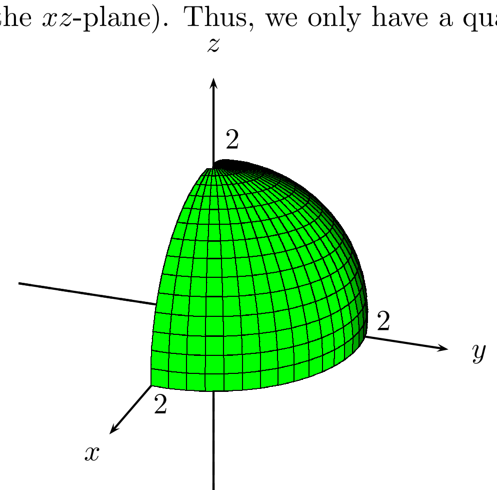
$$I = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} f(x, y, z)\, dx\, dz\, dy.$$

(a) Rewrite the integral in spherical coordinates. *Do not evaluate this integral (yet).*

Solution: The limits of integration give,

$$-\sqrt{4 - y^2 - z^2} \leq x \leq \sqrt{4 - y^2 - z^2}, \quad 0 \leq z \leq \sqrt{4 - y^2}, \quad 0 \leq y \leq 2,$$

which gives part of the sphere, $x^2 + y^2 + z^2 = 4$. The region is bounded by $z = 0$ (the xy-plane) and $y = 0$ (the xz-plane). Thus, we only have a quarter of the sphere, shown in the following figure:



In spherical coordinates, this corresponds to limits of

$$0 \leq \rho \leq 2, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \pi.$$

Also, $f(x, y, z) = x^2 + y^2 + z^2 = \rho^2$. Thus, the integral becomes

$$I = \int_0^{\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^2 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = \int_0^{\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^4 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

(b) Evaluate **either** the original integral or the one in part (a).

Solution: Using spherical coordinates is much easier. Thus,

$$I = \int_0^\pi \int_0^{\frac{\pi}{2}} \frac{1}{5}\rho^5 \sin(\phi)\Big|_0^2 \, d\phi\, d\theta = \int_0^\pi \int_0^{\frac{\pi}{2}} \frac{32}{5} \sin(\phi)\, d\phi\, d\theta.$$

$$= \int_0^\pi -\frac{32}{5}\cos(\phi)\Big|_0^{\frac{\pi}{2}} d\theta = \int_0^\pi -\frac{32}{5}(0-1)\ d\theta = \int_0^\pi \frac{32}{5}\, d\theta = \frac{32}{5}\pi.$$

5. (10 points) Let $\mathbf{F}(x,y) = \langle 2x, -y\rangle$.

(a) Is $\mathbf{F}$ a conservative field? If so, find a potential function for $\mathbf{F}$.

Solution: Since our vector is in 2D, we can check the 2D curl,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - 0 = 0,$$

Therefore, $\mathbf{F}$ is conservative and $\mathbf{F} = \nabla\phi$. To find the potential function, $\phi(x,y)$, we know,

$$\frac{\partial \phi}{\partial x} = 2x \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -y.$$

We first integrate in x,

$$\phi(x,y) = \int 2x\, dx = x^2 + g(y),$$

where g(y) is a constant of integration for x. Taking the y derivative yields,

$$\frac{\partial \phi}{\partial y} = -y = g'(y).$$

Then, integrating in y yields,

$$g(y) = \int -y\, dy = -\frac{y^2}{2} + c,$$

where c is an actual constant. Thus, $\phi(x,y) = x^2 - \frac{y^2}{2} + c$.

(b) Let $\mathcal{C}$ be the curve parameterized by $\mathbf{r}(t) = \langle \sin(t), t\rangle$, for $0 \le t \le \frac{\pi}{2}$. Evaluate $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

Solution: Since, $\mathbf{F}$ is conservative, we know the line integral of it can be computed only using the endpoints of the curve and the potential function. Therefore,

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \phi(\mathbf{r}(\frac{\pi}{2})) - \phi(\mathbf{r}(0)).$$

$$\mathbf{r}(\frac{\pi}{2})) = \langle \sin(\frac{\pi}{2}), \frac{\pi}{2}\rangle = \langle 1, \frac{\pi}{2}\rangle.$$

$$\mathbf{r}(0) = \langle 0, 0\rangle.$$

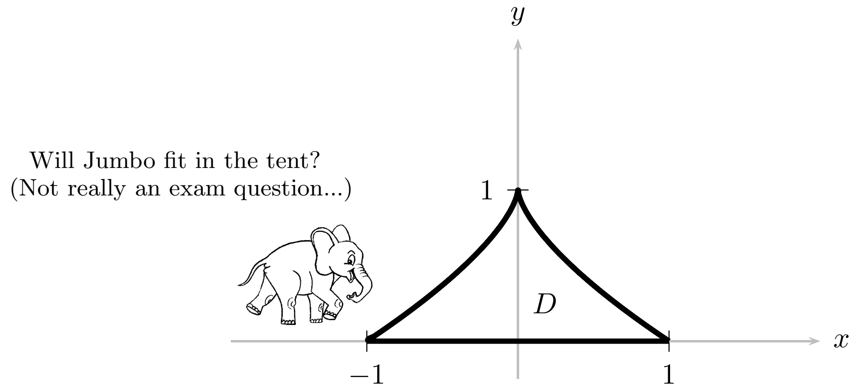
Using part (a), we get,

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \phi(1, \frac{\pi}{2}) - \phi(0,0) = \left(1 - \frac{1}{2}\frac{\pi^2}{4} + c\right) - (0 - 0 + c) = 1 - \frac{\pi^2}{8}.$$

6. (15 points) Using Green's Theorem, compute the area of the region, D, between the x-axis and the tent-shaped curve parametrized as follows:

$$\mathbf{r}(t) = \langle \cos^3(t), \sin^2(t) \rangle \quad \text{for} \;\; 0 \le t \le \pi.$$

Make sure to explicitly show how you are using Green's Theorem to get full credit. A figure of the region D is shown below:



Solution: Green's Theorem states that if $\mathcal{C}$ is the boundary curve of the region D, we have:

$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

We can use this to compute the area of the region, D, if $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = 1$. There are many ways to choose P and Q, one being $P = 0$ and $Q = x$. For this problem, this leads to this easiest integral to compute. Therefore,

$$\text{Area} = \iint_D dA = \oint_{\mathcal{C}} x\,dy.$$

The boundary curve is made up of two curves, the tent-shaped curve we'll call $\mathcal{C}_1$ and the x-axis we'll call $\mathcal{C}_2$. Notice that plugging in values for the tent-shaped curve has it's trajectory going counter-clockwise and is, thus, the positive orientation. Then,

$$\text{Area} = \int_{\mathcal{C}_1} \langle 0, x \rangle \cdot d\mathbf{r}_1 + \int_{\mathcal{C}_2} \langle 0, x \rangle \cdot d\mathbf{r}_2.$$

For $\mathcal{C}_1$ we have the given curve,

$$\mathbf{r}_1(t) = \langle \cos^3(t), \sin^2(t) \rangle \quad \text{for} \;\; 0 \le t \le \pi.$$

$$\mathbf{r}_1'(t) = \langle -3\cos^2(t)\sin(t), 2\sin(t)\cos(t),$$

$$\langle 0, x(t) \rangle = \langle 0, \cos^3(t) \rangle$$

$$\Rightarrow \int_{\mathcal{C}_1} \langle 0, x \rangle \cdot d\mathbf{r}_1 = \int_0^{\pi} 2\cos^4(t)\sin(t)\,dt$$

Using the u-substitution, $u = \cos(t)$, $du = -\sin(t)\,dt$, we get:

$$\int_1^{-1} -2u^4\,dt = -\,2\frac{u^5}{5}\bigg|_1^{-1} = \frac{4}{5}.$$

For the second curve, $\mathcal{C}_2$, we parametrize the x-axis from -1 to 1 as

$$\mathbf{r}_2(t) = \langle t, 0\rangle \qquad -1 \le t \le 1.$$

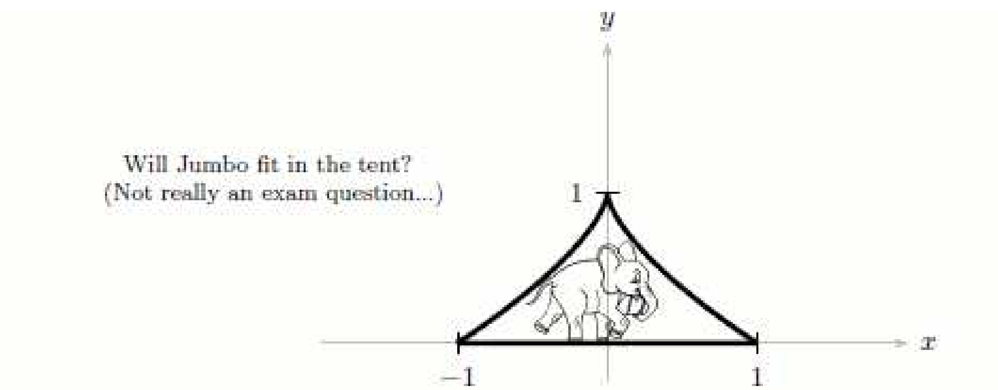
$$\mathbf{r}_2'(t) = \langle 1, 0\rangle.$$

$$\int_{\mathcal{C}_2} \langle 0, x\rangle \cdot d\mathbf{r_2} = \int_{-1}^{1} 0\,dt = 0.$$

Thus,

$$\text{Area} = \frac{4}{5} + 0 = \frac{4}{5}.$$

Oh and by the way...he does fit!

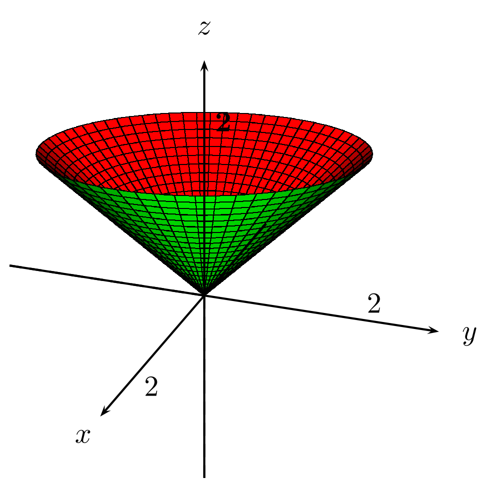


7. (10 points) Let S be the surface parametrized by

$$\mathbf{t}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r\rangle, \qquad 0 \le \theta \le 2\pi, \quad 0 \le r \le 2.$$

(a) Sketch the surface.

Solution: The parametric surface represents the cone $z = r$ up to the plane $z = 2$.



(b) Evaluate $\iint\limits_S z\,dS$

Solution: The surface integral becomes,

$$\iint\limits_S z\,dS = \iint\limits_D z(r,\theta)\,|\mathbf{t}_r(r,\theta)\times\mathbf{t}_\theta(r,\theta)|\,dA,$$

where D is the domain of the parameters r and θ.

$$\mathbf{t}_r = \langle\cos(\theta),\sin(\theta),1\rangle \qquad \mathbf{t}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle.$$

$$\mathbf{t}_r\times\mathbf{t}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & 1 \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \langle -r\cos(\theta), -r\sin(\theta), r\rangle.$$

$$|\mathbf{t}_r\times\mathbf{t}_\theta| = \sqrt{r^2\cos^2(\theta)+r^2\sin^2(\theta)+r^2} = \sqrt{2}r.$$

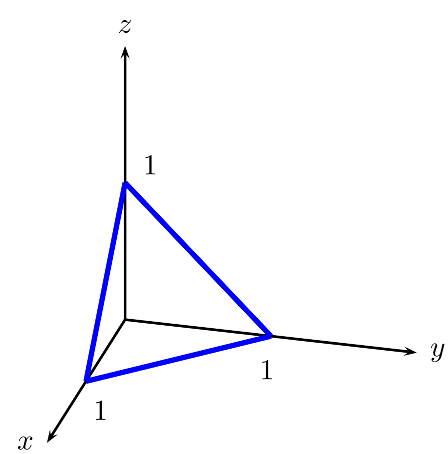
$$\Rightarrow \iint\limits_S z\,dS = \int_0^{2\pi}\int_0^2 r\sqrt{2}r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1}{3}r^3\Big|_0^2\,d\theta = \frac{8\sqrt{2}}{3}\int_0^{2\pi} d\theta = \frac{16\sqrt{2}\pi}{3}.$$

8. (15 points) Let $\mathbf{F} = \langle y^2, z^2, x^2\rangle$ and let $\mathcal{C}$ be the curve that bounds the triangular plate, $x+y+z=1$, in the first octant ($x>0$, $y>0$, $z>0$), oriented clockwise when viewed from above.

   Use Stokes' Theorem to compute the circulation, $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

   Solution: Notice that the partial derivatives of the components of $\mathbf{F}$ are continuous everywhere, so it is safe to use Stokes' Theorem on this vector field. The triangular plate is shown in the following figure:



   If we are looking from above, the positive orientation would yield a normal vector pointing out of the plate (up in z). Thus, the positive orientation of the boundary curve should be counterclockwise. Therefore, we need to reverse the sign of Stokes' Theorem or use the opposite normal vector.

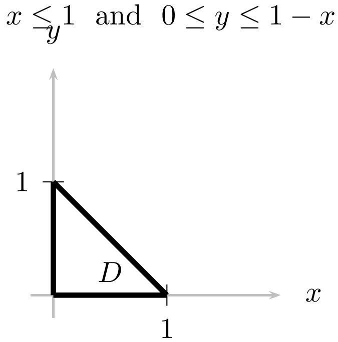
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = -\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot d\mathcal{S},$$

where $\mathcal{S}$ is the surface of the plate with upward normal vector. We parametrize $\mathcal{S}$ as follows:

$$\mathbf{t}(x, y) = \langle x, y, 1 - x - y \rangle,$$

where the domain, D, of the parameters x and y is the triangular region:

$$0 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq 1 - x.$$



Thus,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = -\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot d\mathcal{S} = -\iint_{D} (\nabla \times \mathbf{F})(\mathbf{t}(x, y)) \cdot (\mathbf{t}_x \times \mathbf{t}_y)\, dA,$$

making sure we have the appropriate sign.

$$\nabla \times \mathbf{F} = \langle -2z, -2x, -2y \rangle.$$

$$\nabla \times \mathbf{F}(\mathbf{t}(x, y)) = \langle -2(1 - x - y), -2x, -2y \rangle.$$

$$\mathbf{t}_x = \langle 1, 0, -1 \rangle \qquad \mathbf{t}_y = \langle 0, 1, -1 \rangle.$$

$$\mathbf{t}_x \times \mathbf{t}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle.$$

This is the positive direction as it points up and out. Therefore,

$$-\iint_{D} (\nabla \times \mathbf{F})(\mathbf{t}(x, y)) \cdot (\mathbf{t}_x \times \mathbf{t}_y)\, dA = -\int_0^1 \int_0^{1-x} \langle -2 + 2x + 2y, -2x, -2y \rangle \cdot \langle 1, 1, 1 \rangle \, dy\, dx.$$

$$= -\int_0^1 \int_0^{1-x} (-2 + 2x + 2y - 2x - 2y)\; dy\, dx = -\int_0^1 \int_0^{1-x} -2\, dy\, dx$$

$$= 2\int_0^1 (1 - x)\, dx = -\,(1 - x)^2\Big|_0^1 = 1.$$

9. (10 points) Let $\mathbf{F}(x, y, z) = \langle x^2, y, x - z\rangle$ and S be the boundary surfaces of the region contained in the cylinder $x^2 + y^2 = 1$ between the planes $z = y$ and $z = 2$, with outward unit normal.

Evaluate $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$.

Solution: Since $\mathbf{F}$ has continuous partial derivatives everywhere and S is a smooth closed boundary surface to a volume, V, we can use the divergence theorem,

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iiint\limits_V (\nabla \cdot \mathbf{F})\, dV.$$

The volume, V, is the cylinder between the planes $z = y$ and $z = 2$, therefore, we will use cylindrical coordinates to integrate the region and slice in z first. The plane $z = y$ becomes $z = r\sin(\theta)$ and the cylinder is $r = 1$. Therefore, the region of integration is

$$r\sin(\theta) \le z \le 2, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

Next, we compute the divergence of $\mathbf{F}$,

$$\nabla \cdot \mathbf{F} = 2x + 1 - 1 = 2x,$$

which equals, $2r\cos(\theta)$ in cylindrical coordinates. Thus,

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 \int_{r\sin(\theta)}^2 (2r\cos(\theta))\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(2r^2\cos(\theta)\right)(2 - r\sin(\theta))\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{4}{3}r^3\cos(\theta) - \frac{1}{2}r^4\cos(\theta)\sin(\theta)\right]_0^1 d\theta = \int_0^{2\pi} \left(\frac{4}{3}\cos(\theta) - \frac{1}{2}\cos(\theta)\sin(\theta)\right) d\theta$$

$$= \left[\frac{4}{3}\sin(\theta) - \frac{1}{4}\sin^2(\theta)\right]_0^{2\pi} = 0.$$

End of Exam

According to a recent survey, 100% of all people say they participate in surveys.