Tufts University

Math 13 | **Department of Mathematics** | **May 7, 2012**

Final Exam | **8:30 am to 10:30 am**

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (10 points) **True or False - No Partial Credit**: On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be three nonzero vectors in $\mathbb{R}^3$. Then, $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$.

   (b) Let $g(x, y)$ be a continuous function defined on the domain $D = \{(x, y) \mid a \le x \le b,\ c \le y \le d\}$. The integral
   $$\iint_D g(x,y)dA = \left(\int_a^b g(x,y)\,dx\right)\left(\int_c^d g(x,y)\,dy\right).$$

   (c) Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

   (d) Let $\mathbf{F} = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z\rangle$. Stokes' Theorem says that $\iint\limits_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ for $\mathcal{C}$ being the closed boundary curve of the top half of the unit sphere, S.

   (e) Let $\nabla \cdot \mathbf{F} = 1$ and let S be the surface of the unit cube, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, with outward unit normal. Then, $\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = 1$.

2. (10 points) Let
   $$\mathbf{r}(t) = (2+t)\ \mathbf{i} + (t^2+3)\ \mathbf{j} + (\frac{2}{3}t^3 - 5)\ \mathbf{k}.$$

   (a) Find the parametric equations for the tangent line to the curve $\mathbf{r}(t)$ at $t = 3$.

   (b) Find the arc length of $\mathbf{r}(t)$ from $t = 0$ to $t = 3$.

3. (10 points) Find the absolute minimum and maximum for the function $f(x, y) = 2x + 2y^2 + 1$ on the unit disk $D = \{(x, y) \mid x^2 + y^2 \le 1\}$.

4. (10 points) Let $f(x, y, z) = x^2 + y^2 + z^2$, and consider the integral
   $$I = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} f(x,y,z)\,dx\,dz\,dy.$$

   (a) Rewrite the integral in spherical coordinates. *Do not evaluate this integral (yet).*

   (b) Evaluate **either** the original integral or the one in part (a).

The exam continues on the back!

5. (10 points) Let $\mathbf{F}(x, y) = \langle 2x, -y \rangle$.

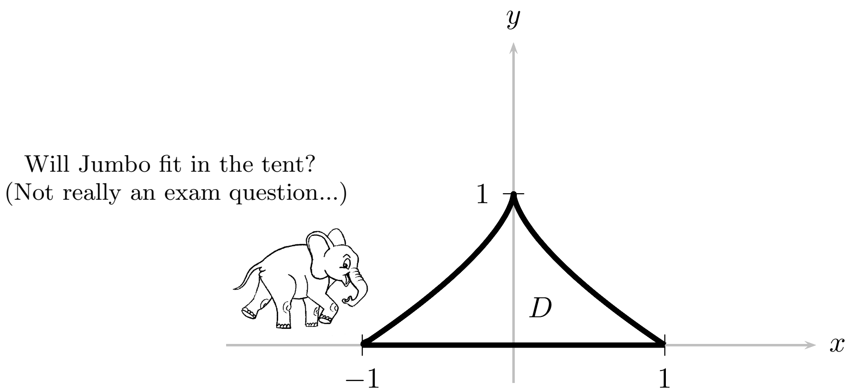
   (a) Is $\mathbf{F}$ a conservative field? If so, find a potential function for $\mathbf{F}$.

   (b) Let $\mathcal{C}$ be the curve parameterized by $\mathbf{r}(t) = \langle \sin(t), t \rangle$, for $0 \le t \le \frac{\pi}{2}$. Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

6. (15 points) Using Green's Theorem, compute the area of the region, D, between the x-axis and the tent-shaped curve parametrized as follows:

$$\mathbf{r}(t) = \langle \cos^3(t), \sin^2(t) \rangle \quad \text{for} \quad 0 \le t \le \pi.$$

   Make sure to explicitly show how you are using Green's Theorem to get full credit. A figure of the region D is shown below:



7. (10 points) Let S be the surface parametrized by

$$\mathbf{t}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), r \rangle, \qquad 0 \le \theta \le 2\pi, \quad 0 \le r \le 2.$$

   (a) Sketch the surface.

   (b) Evaluate $\iint\limits_S z\, dS$

8. (15 points) Let $\mathbf{F} = \langle y^2, z^2, x^2 \rangle$ and let $\mathcal{C}$ be the curve that bounds the triangular plate, $x + y + z = 1$, in the first octant ($x > 0$, $y > 0$, $z > 0$), oriented clockwise when viewed from above.

   Use Stokes' Theorem to compute the circulation, $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

9. (10 points) Let $\mathbf{F}(x, y, z) = \langle x^2, y, x - z \rangle$ and S be the boundary surfaces of the region contained in the cylinder $x^2 + y^2 = 1$ between the planes $z = y$ and $z = 2$, with outward unit normal.

   Evaluate $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$.

End of Exam

According to a recent survey, 100% of all people say they participate in surveys.