1. (10 points) True or False - No Partial Credit: On the first page of your blue book, answer the following questions as True or False.

(a) If \( D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \), then the area of \( D \) is given by \( \int_{g_1(x)}^{g_2(x)} \int_{a}^{b} 1 \, dx \, dy \).

**Solution:** False. The order of integration is incorrect.

(b) The integral \( \int_{0}^{2\pi} \int_{0}^{2} \int_{r}^{2} dz \, dr \, d\theta \) represents the volume enclosed by the cone \( z = \sqrt{x^2 + y^2} \) and the plane \( z = 2 \).

**Solution:** False. In cylindrical coordinates, \( dV = r \, dz \, dr \, d\theta \). The \( r \) is missing here.

(c) The sphere, \( x^2 + y^2 + z^2 = 2x \), in Cartesian coordinates is represented as \( \rho = 2 \cos(\theta) \sin(\phi) \) in Spherical coordinates.

**Solution:** True. Converting to spherical we get, \( \rho = 2 \cos(\theta) \sin(\phi) \).

(d) Let \( \mathbf{F}(x, y) \) be conservative. If \( C \) is the unit circle, then \( \int_{C} \mathbf{F}(x, y) \cdot d\mathbf{r} = 0 \).

**Solution:** True. Line integrals of conservative vector fields are independent of path and, therefore, on a closed path, the integral is 0.

(e) The vector field \( \mathbf{F}(x, y, z) = \langle x, y, z \rangle \) is conservative on \( \mathbb{R}^3 \).

**Solution:** True. We check the following relationships for \( \mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle \).

\[
\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \Rightarrow \frac{\partial(z)}{\partial y} - \frac{\partial(y)}{\partial z} = 0 - 0 = 0,
\]

\[
\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \Rightarrow \frac{\partial(x)}{\partial z} - \frac{\partial(z)}{\partial x} = 0 - 0 = 0,
\]

\[
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \frac{\partial(y)}{\partial x} - \frac{\partial(x)}{\partial y} = 0 - 0 = 0.
\]

Therefore, it is conservative.
2. (10 points) Evaluate the following integrals. You may use any transformations or integration rules that you wish, but you must explain all of your work.

(a) \[ \iint_R 2x \cos(x^2) \, dA, \quad R = \{(x, y) \mid 0 \leq x \leq \sqrt{\pi}, \, 0 \leq y \leq \pi\}. \]

\textbf{Solution:}
\[ \int_0^\pi \int_0^{\sqrt{\pi}} 2x \cos(x^2) \, dx \, dy = \int_0^\pi \left[ \sin(x^2) \right]_0^{\sqrt{\pi}} \, dx, \]
where a \( u \)-substitution \( u = x^2 \) was used.

\[ \int_0^\pi 2x \cos(x^2) \, dA = \int_0^\pi \left( \sin(\pi) - \sin(0) \right) \, dx = \int_0^\pi 0 \, dy = 0. \]

(b) \[ \int_0^\pi \int_y^\pi \cos(x^2) \, dx \, dy. \]

\textbf{Solution:} We need to switch the order of integration. According to the integrands, \( y \leq x \leq \pi \) and \( 0 \leq x \leq \pi \), which gives the picture below:

\[ \begin{array}{c}
\text{The region of integration is the lower triangle so } 0 \leq y \leq x \text{ and } 0 \leq y \leq \pi, \text{ which gives,} \\
\int_0^\pi \int_0^x \cos(x^2) \, dy \, dx = \int_0^\pi \left[ y \cos(x^2) \right]_0^x \, dx = \int_0^\pi x \cos(x^2) \, dx. \\
\text{Using a } u \text{-substitution, } u = x^2, \text{ one gets,} \\
\frac{1}{2} \left[ \sin(x^2) \right]_0^\pi = \frac{1}{2} \sin(\pi^2). \end{array} \]
3. (10 points) Consider the integral \( \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2x y^2 \, dy \, dx \).

(a) Sketch the region of integration.

Solution:

(b) Rewrite the integral in polar coordinates.

Solution:

\[
\int \int_D 2xy^2 \, dA = \int_{-\pi/2}^{\pi/2} \int_0^1 r(2r \cos(\theta))(r^2 \sin^2(\theta)) \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^1 2r^4 \cos(\theta) \sin^2(\theta) \, dr \, d\theta.
\]

(c) Evaluate the integral, in either Cartesian or polar coordinates.

Solution: We’ll use polar coordinates:

\[
\int_{-\pi/2}^{\pi/2} \int_0^1 2r^4 \cos(\theta) \sin^2(\theta) \, dr \, d\theta = \frac{2}{5} \int_{-\pi/2}^{\pi/2} \left[ r^5 \cos(\theta) \sin^2(\theta) \right]_0^1 \, d\theta = \frac{2}{5} \int_{-\pi/2}^{\pi/2} \cos(\theta) \sin^2(\theta) \, d\theta.
\]

Let \( u = \sin(\theta) \), then \( du = \cos(\theta) d\theta \). and we get,

\[
\frac{2}{5} \int_{-1}^{1} u^2 \, du = \frac{2}{15} [u^3]_{-1}^{1} = \frac{2}{15} (1 - (-1)) = \frac{4}{15}.
\]
4. (10 points) Let V be the solid region bounded by \( 4z = x^2 + y^2 \) and \( z = 4 \). Express \( \iiint f(x, y, z) \, dV \) in the following two orders: \( dz \, dy \, dx \) and \( dy \, dx \, dz \).

**Do not evaluate the integrals.**

**Solution:** The region is given in the following picture:

Slicing this in \( z \) first yields the bounds, \( \frac{x^2 + y^2}{4} \leq z \leq 4 \). The shadow or projection of these slices on the \( xy \)-plane is, then, the circle of radius 4:

For this circle, we can slice in \( y \) first, then \( x \), \( -\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2} \) and \( -4 \leq x \leq 4 \). Thus, the integral is:

\[
\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\frac{x^2+y^2}{4}}^{4} f(x, y, z) \, dz \, dy \, dx.
\]
For the $dy \, dx \, dz$ order, we slice in $y$ first, which is bounded between,
\[-\sqrt{4z-x^2} \leq y \leq \sqrt{4z-x^2}.\] The projection on the $xz$-plane is the parabola at $y = 0$, $z = \frac{1}{4}x^2$:

Next, we slice this 2D region in $x$ first, 
\[-\sqrt{4z} \leq x \leq \sqrt{4z} \text{ and } 0 \leq z \leq 4.\] Thus, the integral becomes,
\[
\int_{0}^{4} \int_{-\sqrt{4z}}^{\sqrt{4z}} \int_{-\sqrt{4z-x^2}}^{\sqrt{4z-x^2}} f(x, y, z) \, dy \, dx \, dz.
\]
5. (10 points) Compute \( \iiint_V \frac{1}{x^2 + y^2 + z^2} \, dV \), where \( V \) is the solid region between two concentric spheres centered at the origin, \( x^2 + y^2 + z^2 = 1 \) and \( x^2 + y^2 + z^2 = 4 \).

**Solution:** Using spherical coordinates, we can rewrite the two spheres as, \( \rho = 1 \) and \( \rho = 2 \). Thus, the region of integration is the region between the two spheres, so 

\[ 1 \leq \rho \leq 2, \quad 0 \leq \phi \leq \pi, \quad \text{and} \quad 0 \leq \theta \leq 2\pi. \]

Thus, 1 \( \times \) \( x^2 + y^2 + z^2 \rightarrow \rho \) in spherical coordinates. Thus,

\[
\iiint_V \frac{1}{x^2 + y^2 + z^2} \, dV = \int_0^{2\pi} \int_0^\pi \int_1^2 \frac{1}{\rho^2} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^\pi \int_1^2 \sin(\phi) \, d\rho \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \left[ -\cos(\phi) \right]_0^\pi \, d\theta = \int_0^{2\pi} (1 + 1) \, d\theta = 4\pi.
\]

6. (10 points) Let \( \mathbf{F}(x, y, z) = \langle y, 5z, 4x \rangle \). Compute \( \int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} \), where \( C \) is defined by \( r(t) = \langle t, t^2, t^3 \rangle \quad 0 \leq t \leq 1 \).

**Solution:** We check the partial derivatives of \( \mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle \).

\[
\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \Rightarrow \frac{\partial (4x)}{\partial y} - \frac{\partial (5z)}{\partial z} = 0 - 5 = -5,
\]

\[
\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \Rightarrow \frac{\partial (y)}{\partial z} - \frac{\partial (4x)}{\partial x} = 0 - 5 = -4,
\]

\[
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \frac{\partial (5z)}{\partial x} - \frac{\partial (y)}{\partial y} = 0 - 1 = -1.
\]

Therefore, \( \mathbf{F} \) is not conservative.

We then use the formula for calculating the line integral of a vector field,

\[
\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(r(t)) \cdot r'(t) \, dt
\]

First, we need \( r'(t) \), then \( \mathbf{F} \) evaluated along the curve, and then the two vectors’ dot product:

\[
r'(t) = \langle 1, 2t, 3t^2 \rangle,
\]

\[
\mathbf{F}(r(t)) \cdot r'(t) = \langle t^2, 5t^3, 4t \rangle \cdot \langle 1, 2t, 3t^2 \rangle = t^2 + 10t^4 + 12t^3.
\]

\[
\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \int_0^1 \left( t^2 + 10t^4 + 12t^3 \right) dt = \left[ \frac{1}{3} t^3 + 2t^5 + 3t^4 \right]_0^1
\]

\[
= \frac{1}{3} + 2 + 3 = \frac{16}{3}.
\]
7. (15 points) Consider \( f(x, y, z) = x + y + z \).

(a) Find the line integral of \( f(x, y, z) \) over the line segment from the point \((1, 2, 3)\) to \((-1, 4, 4)\).

**Solution:** First we find the vector function of the curve. A vector parallel to the line segment is: \( v = (-1 - 1, 4 - 2, 4 - 3) = (-2, 2, 1) \). Thus, the curve can be written as

\[
C : \mathbf{r}(t) = (1 - 2t, 2 + 2t, 3 + t) \quad 0 \leq t \leq 1
\]

Then,

\[
\int_C x + y + z \, ds = \int_0^1 ((1 - 2t) + (2 + 2t) + (3 + t)) |\mathbf{r}'(t)| dt.
\]

\[
\mathbf{r}'(t) = (-2, 2, 1) \Rightarrow |\mathbf{r}'(t)| = \sqrt{4 + 4 + 1} = 3.
\]

\[
\int_C f(x, y, z) \, ds = \int_0^1 3(6 + t) \, dt = \left[ 18 + \frac{3}{2} t^2 \right]_0^1 = 18 + \frac{3}{2} = \frac{39}{2}
\]

(b) What is the line integral of \( \nabla f(x, y, z) \) over the same line segment?

**Solution:** From the fundamental theorem of line integrals:

\[
\int_C \nabla f(x, y, z) \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(-1, 4, 4) - f(1, 2, 3) = (-1 + 4 + 4) - (1 + 2 + 3) = 7 - 6 = 1.
\]

8. (15 points) Let \( \mathbf{F}(x, y, z) = (2xy + z, x^2 + \cos(y), x) \). Find the potential function of \( \mathbf{F}(x, y, z) \) if it exists, or show that one does not exist.

**Solution:** First quickly check that \( \mathbf{F} = (P(x, y, z), Q(x, y, z), R(x, y, z)) \) is conservative:

\[
\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \Rightarrow \frac{\partial(x)}{\partial y} - \frac{\partial(x^2 + \cos(y))}{\partial z} = 0 - 0 = 0,
\]

\[
\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \Rightarrow \frac{\partial(2xy + z)}{\partial z} - \frac{\partial(x)}{\partial x} = 1 - 1 = 0,
\]

\[
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \frac{\partial(x^2 + \cos(y))}{\partial x} - \frac{\partial(2xy + z)}{\partial y} = 2x - 2x = 0.
\]

Therefore, \( \mathbf{F} \) is conservative and we know there exists a potential function, \( \phi(x, y, z) \) such that \( \mathbf{F} = \nabla \phi \).

\[
\phi_x = 2xy + z
\]

Integrate with respect to \( x \):

\[
\Rightarrow \phi(x, y, z) = \int (2xy + z) \, dx = x^2y + zx + g(y, z)
\]

\[
\phi_y = x^2 + \cos(y) = x^2 + g_y.
\]

\[
\Rightarrow g_y = \cos(y).
\]

Integrate with respect to \( y \),

\[
g(y, z) = \int \cos(y) \, dy = \sin(y) + h(z).
\]
\[ \Rightarrow \phi(x, y, z) = x^2y + zx + \sin(y) + h(z). \]
\[ \phi_z = x = x + h'(z). \]
\[ \Rightarrow h'(z) = 0. \]

Integrate with respect to \( z \),
\[ h(z) = \int 0 \, dz = c, \]
where \( c \) is a constant.
\[ \Rightarrow \phi(x, y, z) = x^2y + zx + \sin(y) + c. \]

Quick check:
\[ \nabla \phi = \langle 2xy + z, x^2 + \cos(y), x \rangle. \]

9. (10 points) Consider the solid \( E \) bounded by the sphere \( x^2 + y^2 + z^2 = 2z \) and the sphere \( x^2 + y^2 + z^2 = 1 \). Set up and evaluate a triple integral in Cylindrical Coordinates to calculate the volume of \( E \).

**Solution:** (Note: As written the question is ambiguous to which domain we are referring to. We intended the question to refer to the region between the bottom of the sphere centered at (0,0,1) (sphere 1) and the top of the sphere centered at the origin (sphere 2). However, we will also accept answers for the volume of the region between the top of sphere 1 and the bottom of sphere 2, or between the bottom of sphere 1 and the top of sphere 2. We will show solutions for all 3 regions.)

**Region between bottom of sphere 1 and top of sphere 2**

Notice that the first sphere, is a sphere of radius 1 shifted up the \( z \)-axis, centered at (0,0,1). We obtain this by completing the square:
\[ x^2 + y^2 + z^2 - 2z + 1 - 1 = 0 \Rightarrow x^2 + y^2 + (z - 1)^2 = 1. \]

Thus, the solid \( E \) is bounded between the upper half of the unit sphere, \( x^2 + y^2 + z^2 = 1 \) and the lower half of the shifted sphere, \( x^2 + y^2 + z^2 = 2z \):
Even though, the region is between two spheres, it is much easier to integrate with respect to cylindrical coordinates. If we slice in $z$ first, the region is between the two spheres, so

$$1 - \sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{1 - x^2 - y^2},$$

or in cylindrical coordinates,

$$1 - \sqrt{1 - r^2} \leq z \leq \sqrt{1 - r^2}.$$

The projection on the $xy$-plane is then the circle where the two spheres meet:

$$1 - \sqrt{1 - r^2} = \sqrt{1 - r^2}$$

$$1 = 2\sqrt{1 - r^2}$$

$$\frac{1}{4} = 1 - r^2 \Rightarrow r = \frac{\sqrt{3}}{2}.$$

This is the circle centered at the origin in the $xy$-plane with radius $\frac{\sqrt{3}}{2}$. Thus, the volume integral becomes:

$$\iiint_E dV = \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \int_{\sqrt{1 - r^2}}^{\sqrt{1 - z^2}} rz \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \left( r(\sqrt{1 - r^2}) - r(1 - \sqrt{1 - r^2}) \right) \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} (2r\sqrt{1 - r^2} - r) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[ -\frac{2}{3}(1 - r^2)^{\frac{3}{2}} - \frac{1}{2}r^2 \right]_0^{\frac{\sqrt{3}}{2}} d\theta,$$

where the $u$-substitution, $u = 1 - r^2$ → $du = -2r \, dr$ was used for the 2nd part of the integral.

$$= \int_0^{2\pi} \left( -\frac{2}{3}(1 - \frac{3}{4})^{\frac{3}{2}} - \frac{3}{8} \right) - \left( -\frac{2}{3} \right) \, d\theta$$

$$= \int_0^{2\pi} \frac{5}{24} \, d\theta = \frac{5\pi}{12}.$$

**Alternative Solution:** We could also slice in $r$ first. Notice however, that we would have to split the volume into two regions; the top half of the region, $z^2 + r^2 = 1$, and the bottom half, $(z - 1)^2 + r^2 = 1$. In these regions,

$$0 \leq r \leq \sqrt{1 - z^2} \quad \text{and} \quad 0 \leq r \leq \sqrt{1 - (z - 1)^2}.$$

The plane of intersection in $z$ is when $\sqrt{1 - z^2} = \sqrt{1 - (z - 1)^2}$.

$$\Rightarrow 1 - z^2 = 1 - (z - 1)^2 \Rightarrow z^2 = z^2 - 2z + 1 \Rightarrow z = \frac{1}{2}$$

Thus, the integral becomes

$$V = \int_0^{2\pi} \int_0^{\frac{1}{2}} \int_0^{\sqrt{1-(z-1)^2}} r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_0^{1} \int_0^{\sqrt{1-z^2}} r \, dr \, dz \, d\theta.$$
Using, symmetry, you also only need to compute one of the integrals and double the answer:

\[ V = 2 \int_{0}^{2\pi} \int_{0}^{\frac{1}{2}} \int_{0}^{\sqrt{1-(z-1)^2}} r \, dz \, dr \, d\theta. \]

\[ = 2 \int_{0}^{2\pi} \int_{0}^{\frac{1}{2}} [\sqrt{\frac{r^2}{2}}]_{0}^{\frac{r^2}{2}} d\theta = \int_{0}^{2\pi} \int_{0}^{\frac{1}{2}} (1-(z-1)^2) \, dz \, d\theta. \]

\[ = \int_{0}^{2\pi} \left[ z - \frac{1}{3}(z-1)^3 \right]_{0}^{\frac{1}{2}} d\theta = \int_{0}^{2\pi} \left( \frac{1}{2} + \frac{1}{24} - \frac{1}{3} \right) d\theta = 2\pi \frac{5}{24} = \frac{5\pi}{12}. \]

**Region between top of sphere 1 and top of sphere 2.**

Slicing in \( z \) first is the easiest, but still requires us to split the integral. Notice that the lower sphere is still going out after the intersection. Thus, before the intersection \( z \) cuts through sphere 2 and then sphere 1. After the intersection, \( z \) cuts through the bottom half of sphere 1 and then the top half of sphere 1. Therefore we have:

\[ \sqrt{1-r^2} \leq z \leq 1 + \sqrt{1-r^2} \quad \text{for} \quad 0 \leq r \leq \frac{\sqrt{3}}{2}, \]

where the intersection point of \( r \) was found above in the first solution method, and,

\[ 1 - \sqrt{1-r^2} \leq z \leq 1 + \sqrt{1-r^2} \quad \text{for} \quad \frac{\sqrt{3}}{2} \leq r \leq 1. \]

Thus,

\[ V = \int_{0}^{2\pi} \int_{\sqrt{\frac{3}{2}}}^{\frac{\sqrt{3}}{2}} \int_{1+\sqrt{1-r^2}}^{1} r \, dz \, dr \, d\theta + \int_{0}^{2\pi} \int_{\sqrt{1-\sqrt{1-r^2}}}^{1} \int_{1+\sqrt{1-r^2}}^{1} r \, dz \, dr \, d\theta. \]

\[ = \int_{0}^{2\pi} \int_{\sqrt{\frac{3}{2}}}^{\frac{\sqrt{3}}{2}} [r z]_{1+\sqrt{1-r^2}}^{1} \, dr \, d\theta + \int_{0}^{2\pi} \int_{\sqrt{1-\sqrt{1-r^2}}}^{1} \int_{1+\sqrt{1-r^2}}^{1} r \, dz \, dr \, d\theta \]
\[ = \int_{0}^{2\pi} \int_{\sqrt{\frac{3}{2}}}^{\frac{\sqrt{3}}{2}} r \, dr \, d\theta + \int_{0}^{2\pi} \int_{\sqrt{1-\sqrt{1-r^2}}}^{1} 2r \sqrt{1-r^2} \, dr \, d\theta \]

For the second integral, let \( u = 1-r^2 \) and \( du = -2r \, dr \), then,

\[ V = \int_{0}^{2\pi} \left[ \frac{1}{2} r^2 \right]_{\sqrt{\frac{3}{2}}}^{0} d\theta + \int_{0}^{\frac{1}{2}} \left[ -\frac{2}{3} u^{\frac{3}{2}} \right]_{0}^{\frac{1}{2}} d\theta \]
\[ = \int_{0}^{2\pi} \left[ 2 \cdot \frac{3}{8} \right] d\theta + \int_{0}^{2\pi} \left[ 2 \cdot \frac{1}{8} \right] d\theta \]
\[ = \frac{3\pi}{4} + \frac{\pi}{6} = \frac{11\pi}{12}. \]

Notice, that if we take the volume, of sphere 2, \( \frac{4\pi}{3} \) and subtract this result, \( \frac{4\pi}{3} - \frac{11\pi}{12} = \frac{5\pi}{12} \), we get the original result, for the region, between the two spheres.
Region between bottom of sphere 2 and bottom of sphere 1.
The same approach as the previous solution applies, but with the bottoms of the sphere, to obtain,

\[
V = \int_{0}^{2\pi} \int_{0}^{\sqrt{3}} \int_{0}^{\sqrt{1 - r^2}} r \, dz \, dr \, d\theta + \int_{0}^{2\pi} \int_{\sqrt{3}}^{1} \int_{0}^{\sqrt{1 - r^2}} r \, dz \, dr \, d\theta.
\]

\[
= \int_{0}^{2\pi} \int_{0}^{\sqrt{3}} [rz]_{0}^{\sqrt{1 - r^2}} \, dr \, d\theta + \int_{0}^{2\pi} \int_{\sqrt{3}}^{1} [rz]_{0}^{\sqrt{1 - r^2}} \, dr \, d\theta
\]

\[
= \int_{0}^{2\pi} \int_{0}^{\sqrt{3}} r \left(1 - \sqrt{1 - r^2} + \sqrt{1 - r^2}\right) \, dr \, d\theta + \int_{0}^{2\pi} \int_{\sqrt{3}}^{1} r \left(\sqrt{1 - r^2} + \sqrt{1 - r^2}\right) \, dr \, d\theta
\]

\[
= \int_{0}^{2\pi} \int_{0}^{\sqrt{3}} r \, dr \, d\theta + \int_{0}^{2\pi} \int_{\sqrt{3}}^{1} 2r \sqrt{1 - r^2} \, dr \, d\theta.
\]

For the second integral, let \( u = 1 - r^2 \) and \( du = -2rdr \), then,

\[
V = \int_{0}^{2\pi} \left[\frac{1}{2}r^2\right]_{0}^{\sqrt{3}} d\theta + \int_{0}^{2\pi} \left[-\frac{2}{3}u^{\frac{3}{2}}\right]_{\frac{1}{4}}^{0} d\theta
\]

\[
= \int_{0}^{2\pi} \frac{1}{2} \cdot \frac{3}{4} d\theta + \int_{0}^{2\pi} \frac{2}{3} \cdot \frac{1}{8} d\theta
\]

\[
= \frac{3\pi}{4} + \frac{\pi}{6} = \frac{11\pi}{12}.
\]

**End of Exam**

There are 10 types of people in the world.
Those who know binary and those who don’t.