Tufts University

Math 13 — **Department of Mathematics** — **Februrary 27, 2012**

Exam 1 Solutions — **12:00 pm to 1:20 pm**

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (10 points) **True or False - No Partial Credit**: On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. The magnitude of $\mathbf{u}+\mathbf{v}$ is at least the magnitude of $\mathbf{u}$.
   Solution: False: Take $\mathbf{u}=\mathbf{i}$ and $\mathbf{v}=-\mathbf{i}$. Then $|\mathbf{u}|=1$, but $|\mathbf{u}+\mathbf{v}|=0$.

   (b) The vectors $\mathbf{u}=\langle 3,2,1\rangle$ and $\mathbf{v}=\langle -6,-4,-2\rangle$ are parallel.
   Solution: True: $\mathbf{v}=-2\mathbf{u}$, so the vectors are parallel by definition.

   (c) The vector $\mathbf{w}=\mathbf{i}-\mathbf{j}$ is orthogonal to the vector $\mathbf{u}=\mathbf{i}+\mathbf{j}+\mathbf{k}$.
   Solution: True: $\mathbf{w}\cdot\mathbf{v}=(1)(1)+(-1)(1)+(0)(1)=1-1=0$. Since the dot product is zero, the vectors are orthogonal.

   (d) The directional derivative of $f(x,y)=xy$ in the direction of the vector $\mathbf{u}=\langle 1,1\rangle$ at the point $(1,1)$ is 2.
   Solution: False: $\nabla f(x,y)=\langle y,x\rangle \Rightarrow \nabla f(1,1)=\langle 1,1\rangle$. The directional derivative of f in the direction of $\mathbf{u}$ is $D_{\hat{\mathbf{u}}}f(1,1)=\nabla f(1,1)\cdot\hat{\mathbf{u}}$. But $\hat{\mathbf{u}}=\frac{\mathbf{u}}{|\mathbf{u}|}=\frac{\langle 1,1\rangle}{\sqrt{2}}$.
   $\Rightarrow D_{\hat{\mathbf{u}}}f(1,1)=\langle 1,1\rangle\cdot\langle\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle=\frac{2}{\sqrt{2}}$.

   (e) $\mathbf{u}-\mathbf{proj_v u}$ is orthogonal to $\mathbf{v}$.
   Solution: True: $\mathbf{v}\cdot(\mathbf{u}-\mathbf{proj_v u})=\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\frac{\mathbf{v}}{|\mathbf{v}|}\right)=\mathbf{v}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}\left(\frac{\mathbf{v}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)=$ $\mathbf{v}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}=0$. Therefore, the vectors are orthogonal.

2. (10 points) Consider the triangle that has vertices $(1,1,1)$, $(2,2,1)$, and $(1,2,2)$.

   (a) What is the area of the triangle?
   Solution: Write two edges of the triangle as

$$\mathbf{u}=\langle 2,2,1\rangle-\langle 1,1,1\rangle=\mathbf{i}+\mathbf{j}$$
$$\mathbf{v}=\langle 1,2,2\rangle-\langle 1,1,1\rangle=\mathbf{j}+\mathbf{k}.$$

   The area of the triangle is, then, $A=\frac{1}{2}|\mathbf{u}\times\mathbf{v}|$. So, we first compute the cross product:

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\0&1&1\end{vmatrix}=1\mathbf{i}-1\mathbf{j}+1\mathbf{k}.$$

   The area is then $A=\frac{1}{2}\sqrt{1+1+1}=\frac{\sqrt{3}}{2}$.

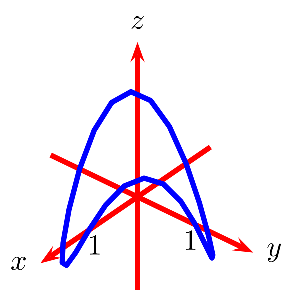
(b) Write an equation of the plane that contains these three points.
Solution: The normal to this plane is the vector $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ computed in part (a). The plane has formula

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = \mathbf{n} \cdot \langle x - 1, y - 1, z - 1 \rangle = 0$$
$$x - y + z = 1.$$

3. (10 points) A roller coaster travels on a track with position given by

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \sin(2t)\mathbf{k} \text{ for } 0 \leq t \leq 2\pi.$$

(a) Plot $\mathbf{r}(t)$ for $0 \leq t \leq 2\pi$.
Solution: Notice that the x and y coordinates go one time counterclockwise around the unit circle as t goes from 0 to 2π. The z coordinate, on the other hand, oscillates more frequently, going from 0 at $t = 0$ to 1 at $t = \frac{\pi}{4}$ to 0 at $t = \frac{\pi}{2}$ and so forth. Plotting points sketches the following shape.



(b) Find the velocity and acceleration of the roller coaster as functions of t.
Solution: By definition,

$$\mathbf{v}(t) = \mathbf{r}'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + 2\cos(2t)\mathbf{k}$$
$$\mathbf{a}(t) = \mathbf{v}'(t) = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j} - 4\sin(2t)\mathbf{k}.$$

(c) Find the unit tangent vector to $\mathbf{r}(t)$ at $t = \frac{\pi}{2}$.
Solution: The unit tangent vector is given by $\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t)$. From part (b),

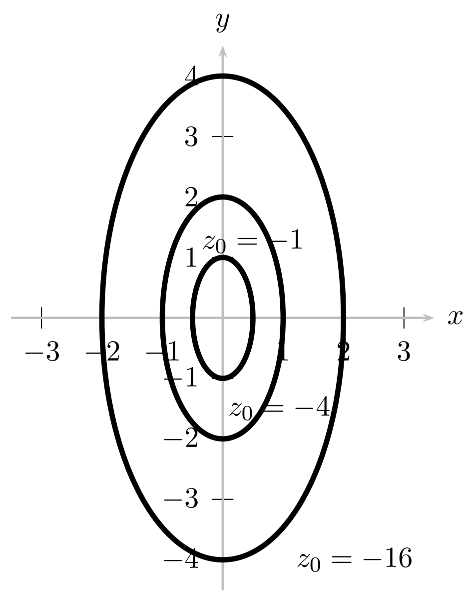
$$|\mathbf{r}'(t)| = \left(\sin^2(t) + \cos^2(t) + 4\cos^2(2t)\right)^{\frac{1}{2}} = \left(1 + 4\cos^2(2t)\right)^{\frac{1}{2}}.$$

So,

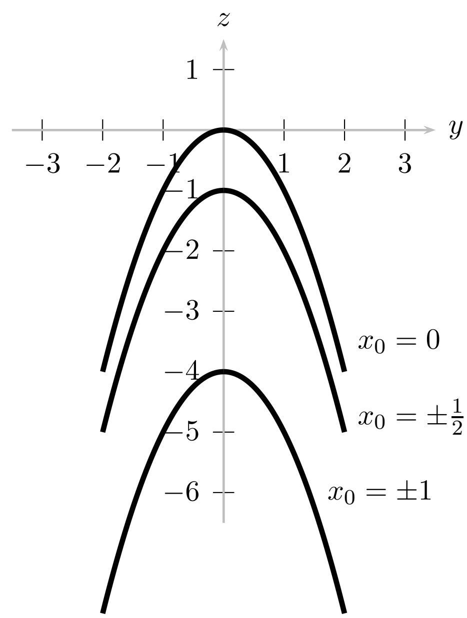
$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{5}}\left(-\mathbf{i} - 2\mathbf{k}\right).$$

4. (15 points) Consider the surface $z = f(x, y) = -4x^2 - y^2$.

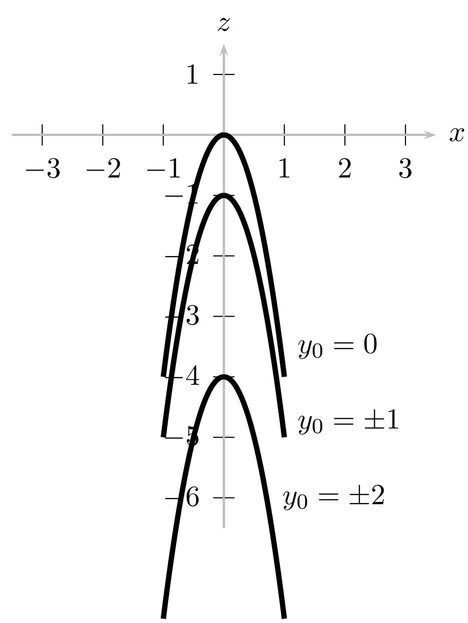
(a) Sketch several level curves of $f(x, y)$, clearly labeling the curves.
Solutions: Level curves are obtained by setting the function equal to a constant, $f(x, y) = -4x^2 - y^2 = z_0$. In this case, they are the xy-traces. Notice, there are no traces if $z_0 > 0$. If $z_0 = 0$, then the trace is a point $(0, 0)$ in the $xy-$plane. If $z_0 < 0$, then the level curve is an ellipse in the $xy-$plane.



(b) Sketch several xz-traces and yz-traces of $z = -4x^2 - y^2$, clearly labeling the traces.
Solution: For the $yz-$ traces, we have a fixed x value, $x = x_0$, and $z = -y^2 - 4x_0^2$. For $x_0 = 0$, $z = -y^2$ gives a parabola through the origin in the $yz-$plane. For other x_0, the parabola goes through $(0, -4x_0^2)$. The traces look like:



Similarly, for the xz-traces, we have a fixed y value, $y = y_0$, and $z = -4x^2 - y_0^2$. For $y_0 = 0$, $z = -4x^2$ gives a parabola through the origin in the $yz-$plane. For other y_0, the parabola goes through $(0, -y_0^2)$. The traces look like:



(c) Give the name for the surface $z = -4x^2 - y^2$.
Solution: We get an upside down bowl-shape, called an **elliptic paraboloid**.

5. (10 points) Let $f(x, y, z)$ be a function in three variables, which satisfies

$$f_x(4,1,2) = 3, \quad f_y(4,1,2) = 7, \quad f_z(4,1,2) = 5.$$

Let

$$x(s,t) = s + 2t, \quad y(s,t) = t^2, \quad z(s,t) = st.$$

Find $\frac{\partial f}{\partial t}$ at the point $(s,t) = (2,1)$.

Solution:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$x(2,1) = 4$, $y(2,1) = 1$, $z(2,1) = 2$.

$$\Rightarrow \frac{\partial f(4,1,2)}{\partial t} = 3(2) + 7(2t) + 5(s) = 6 + 7(2) + 5(2) = 30.$$

6. (10 points) A spaceship is in trouble near the sunny side of Mercury. The temperature (in Celsius) of the spaceship's hull at the point (x, y, z) is given by

$$T(x,y,z) = 10e^{xy+2z},$$

where x, y, and z are all measured in millions of kilometers (Gigameters). The ship is currently at $(1,2,1)$.

(a) In what direction should the spaceship travel in order to experience the fastest *decrease* in temperature? (Express this direction as a unit vector.)

Solution: The fastest increase occurs in the direction of the gradient, therefore, the fastest *decrease* is in the opposite direction: $-\frac{\nabla T}{|\nabla T|}$.

$$\nabla T = \langle 10ye^{xy+2z}, 10xe^{xy+2z}, 20e^{xy+2z}\rangle.$$

$$\nabla T(1,2,1) = 10e^4\langle 2,1,2\rangle.$$

$$|\nabla T(1,2,1)| = 10e^4\sqrt{4+1+4} = 30e^4.$$

The fastest decrease at $(1,2,1)$ is in the direction of:

$$-\frac{\nabla T(1,2,1)}{|\nabla T(1,2,1)|} = \langle -\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\rangle.$$

(b) What is the rate of change of the temperature if the ship proceeds in that direction?

Solution: If the spaceship moves in that direction, it experiences the fastest decrease in temperature which is negative of the magnitude of the gradient as computed above:

$$-|\nabla T(1,2,1)| = -30e^4.$$

7. (10 points) Let $f(x, y) = y\cos(x - y)$.

(a) Find the linear approximation of $f(x, y)$ around the point $(2, 2)$ and use it to estimate $f(2.1, 1.9)$.
Solution: The linear approximation is given by:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

For our function, $f_x = -y\sin(x-y) \Rightarrow f_x(2, 2) = 0$, and $f_y = \cos(x-y)+y\sin(x-y) \Rightarrow f_y(2, 2) = 1$. Also, $f(2, 2) = 2$. Therefore, we get:

$$L(x, y) = 2 + 0(x - 2) + (y - 2) = y.$$

Using this, we estimate,

$$f(2.1, 1.9) \approx L(2.1, 1.9) = 1.9.$$

(b) Write down the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(2, 2)$.
Solution: The tangent plane is the surface described by the linear approximation, namely, $z = L(x, y)$. Therefore, the tangent plane is

$$z = y.$$

8. (10 points) Using Lagrange Multipliers, find the maximum and minimum values of $f(x, y) = x^2 + y^2 - 2y$, subject to the constraint $x^2 + 2y^2 = 8$. **No credit** will be given for a solution that does not make use of Lagrange Multipliers.
Solution: Using Lagrange multipliers we set $f(x, y) = x^2+y^2-2y$ and $g(x, y) = x^2+2y^2 = 8$. Then,

$$\begin{aligned} \nabla f =& \lambda \nabla g \\ 2x =& 2\lambda x \\ 2y - 2 =& 4\lambda y \\ x^2 + 2y^2 =& 8 \end{aligned}$$

Solving the first equation, we get:

$$x = \lambda x.$$

Note, we can not divide by x as it could be zero!!

$$x(1 - \lambda) = 0.$$

This yields two possibilites:

$$x = 0 \quad \text{or} \quad \lambda = 1.$$

For $x = 0$, we use the constraint to get:

$$0^2 + 2y^2 = 8 \rightarrow y = \pm 2.$$

Thus, two possible coordinates are $(0, 2)$ and $(0, -2)$. For $\lambda = 1$, we use the second equation to get:

$$2y - 2 = 4y \Rightarrow -2 = 2y \Rightarrow y = -1.$$
$$\Rightarrow x^2 + 2(-1)^2 = 8 \Rightarrow x = \pm\sqrt{6}.$$

Thus, two other possible coordinates are $(\sqrt{6}, -1)$ and $(-\sqrt{6}, -1)$. Next, we compare the function values,
$f(0, 2) = 4 - 4 = 0$
$f(0, -2) = 4 + 4 = 8$
$f(\pm\sqrt{6}, -1) = 6 + 1 + 2 = 9.$
Thus, the max value of $f(x, y)$ is 9 and occurs at $(\pm\sqrt{6}, -1)$ and the minimum value of $f(x, y)$ is 0 and occurs at $(0, 2)$.

9. (15 points) Find all critical points of the function $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$ and characterize each one, if possible, as a local maximum, a local minimum, or a saddle point.
Solution: Find the partial derivatives and set to zero simultaneously.

$$f_x = 12x - 6x^2 + 6y = 0 \qquad f_y = 6y + 6x = 0.$$

Notice, these are both defined everywhere. The second equation yields,

$$y = -x.$$

Plugging into the first equation gives:

$$12x - 6x^2 - 6x = 6x - 6x^2 = 0.$$

$$x(1 - x) = 0$$

Either $x = 0$ or $x = 1$. If $x = 0$, then $y = 0$ and if $x = 1$, then $y = -1$ from the second equation. Therefore, we have two critical points:
$(0, 0)$ and $(1, -1)$. Next, we apply the second-derivative test to classify the critical points.

$$f_{xx} = 12 - 12x \qquad f_{yy} = 6 \qquad f_{xy} = 6.$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 6(12 - 12x) - 36.$$

For $(0, 0)$:

$$D = 72 - 36 > 0 \qquad f_{xx} = 12 > 0$$

Therefore, $(0, 0)$ is a local minimum.
For $(1, -1)$:

$$D = 0 - 36 < 0.$$

Therefore, $(1, -1)$ is a saddle point.

End of Exam

There are 3 types of people in the world.
Those who can count and those who can't.