Department of Mathematics
Final Exam Solutions

1. (10 points) **True or False.** Write your answers to this problem on the inside front cover of your blue book. Write **T** if the statement is always true, and write **F** otherwise.

   (a) The lines $x = 2 + t,\ y = -1 - t,\ z = -t$ and $x = 1 - t,\ y = 4 + t,\ z = 1 + t$ are parallel.
   True: The first line has direction numbers $1, -1, -1$; the second line has direction numbers $-1, 1, 1$, and these are proportional.

   (b) The area of a parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u} \times \mathbf{v}$.

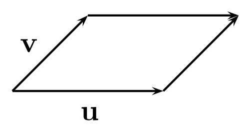

Figure 1: Problem 1b

   False: The area of the parallelogram is $|\mathbf{u} \times \mathbf{v}|$.

   (c) If $f_x = x^2 - y$ and $f_y = y^2 - x$, then $(1, 1)$ is a saddle point of $f(x, y)$.
   False: $f(1, 1)$ is a local minimum.

   (d) The line integral $\int_C \nabla f \cdot d\mathbf{r}$ is independent of path.
   True, by the Fundamental Theorem for Line Integrals, since by definition ∇f is conservative.

   (e) $\int_0^2 \int_x^{\sqrt{8-x^2}} y^2\, dy\, dx = \int_0^{\pi/2} \int_0^{\sqrt{8}} r^3 \sin^2\theta\, dr\, d\theta.$
   False: In fact,

$$\int_0^2 \int_x^{\sqrt{8-x^2}} y^2\, dy\, dx = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{8}} r^3 \sin^2\theta\, dr\, d\theta.$$

2. (7 points) Find the equation of the tangent plane to the parametric surface

$$x = u\cos v,\ y = u\sin v,\ z = v$$

   at the point $(0, 3, \frac{\pi}{2})$.

   Solution: The parametric surface (which is called a *helicoid*, and which looks like a spiral staircase) can also be described by the vector valued function

$$\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}.$$

The point $(x, y, z) = (0, 3, \frac{\pi}{2})$ corresponds to $(u, v) = (3, \frac{\pi}{2})$. Now $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$ and $\mathbf{r}_v = -u\,\sin v\,\mathbf{i} + u\,\cos v\,\mathbf{j} + \mathbf{k}$. Hence

$$\begin{aligned} \mathbf{r}_u(3, \pi/2) &= \mathbf{j} \\ \mathbf{r}_v(3, \pi/2) &= -3\mathbf{i} + \mathbf{k}, \end{aligned}$$

and thus a normal vector to the surface at $(0, 3, \frac{\pi}{2})$ is given by

$$\mathbf{r}_u(3, \pi/2) \times \mathbf{r}_v(3, \pi/2) = \mathbf{i} + 3\,\mathbf{k}.$$

The tangent plane to the surface at $(0, 3, \frac{\pi}{2})$ therefore has equation

$$x + 3(z - \pi/2) = 0,$$

or

$$x + 3z = \frac{3\pi}{2}.$$

3. (8 points) Let $f(x, y) = x^2\,e^{-y}$.

   (a) At the point $(2, 0)$, find the direction in which $f(x, y)$ increases most rapidly. Express the direction as a unit vector.

   Solution: $f(x, y)$ increases most rapidly in the direction of $\nabla f(x, y)$. For $f(x, y) = x^2\,e^{-y}$, we have

   $$\nabla f(x, y) = 2x\,e^{-y}\,\mathbf{i} - x^2\,e^{-y}\,\mathbf{j}$$

   and hence

   $$\nabla f(2, 0) = 4\,\mathbf{i} - 4\,\mathbf{j}.$$

   Therefore the unit vector in the direction of which $f(x, y)$ increases most rapidly at $(2, 0)$ is

   $$\mathbf{v} = \frac{\nabla f(2, 0)}{|\nabla f(2, 0)|} = \frac{4\,\mathbf{i} - 4\,\mathbf{j}}{\sqrt{32}} = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}.$$

   (b) Find a unit vector $\mathbf{u}$ such that the directional derivative $D_{\mathbf{u}} f(2, 0) = 0$.

Solution: Let $\mathbf{w} = \mathbf{i} + \mathbf{j}$. From Part (a), it is obvious that $\nabla f(2, 0) \cdot \mathbf{w} = 0$. If we put

$$\mathbf{u} = \frac{\mathbf{w}}{|\mathbf{w}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}},$$

then $\mathbf{u}$ is a unit vector such that

$$D_{\mathbf{u}}\,f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u} = 0.$$

4. (10 points) Use Lagrange multipliers to find the volume of the largest rectangular box, with edges parallel to the coordinate axes, which can be inscribed in the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$.

Solution: Let $g(x, y, z) = \dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2$. Let (x, y, z) represent the vertex of the rectangular box in the first octant. Since the volume of the box is $V = 8xyz$, we want to maximize the product $f(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = 1$. The vector equation $\nabla f = \lambda \nabla g$ corresponds to the three scalar equations

$$\begin{aligned} yz &= \lambda \frac{x}{2} \\ xz &= \lambda \frac{2y}{9} \\ xy &= \lambda (2z), \end{aligned}$$

and (x,y,z) must also satisfy the equation

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1.$$

The maximum of xyz obviously occurs at a point (x,y,z) where of x, y, and z are all positive. From the equations above, we conclude that $\lambda \neq 0$. Multiplying the first equation in the system above by x, the second equation by y, and the third by z, we obtain

$$xyz = \lambda \frac{x^2}{2} = \lambda \frac{2y^2}{9} = \lambda (2z^2).$$

Since $\lambda \neq 0$, it follows that

$$\frac{x^2}{4} = \frac{y^2}{9} = z^2.$$

Thus the three terms on the left hand side of $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$ are all equal. This results in:

$$\begin{aligned} 3\,\frac{x^2}{4} = 1 &\implies x = \frac{2}{\sqrt{3}} \\ 3\,\frac{y^2}{9} = 1 &\implies y = \sqrt{3} \\ 3\,z^2 = 1 &\implies z = \frac{1}{\sqrt{3}}. \end{aligned}$$

The maximum volume is therefore

$$8\left(\frac{2}{\sqrt{3}}\right)(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right) = \frac{16}{\sqrt{3}}$$

5. (15 points)

(a) The figure below shows the region of integration for the integral

$$\int_0^1 \int_x^1 \int_0^{1-y^2} f(x,y,z)\,dz\,dy\,dx.$$

Rewrite this integral as an equivalent iterated integral in the order $dy\,dz\,dx$.

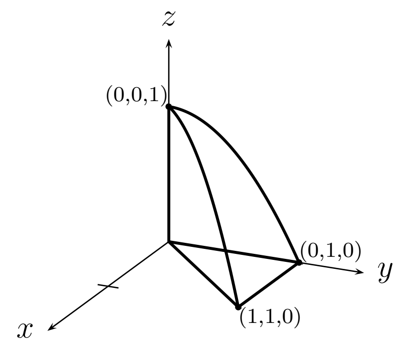

Figure 2: Problem 5a

Solution:

$$\int_0^1 \int_0^{1-x^2} \int_x^{\sqrt{1-z}} f(x,y,z)\,dy\,dz\,dx.$$

(b) Let E be the solid below the sphere $x^2+y^2+z^2=25$ and above the plane $z=4$. Express the volume of E as an iterated triple integral in cylindrical coordinates. **Do not evaluate.**

Solution:

$$\int_0^{2\pi} \int_0^3 \int_4^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta.$$

(c) Convert the triple integral

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2\,dz\,dy\,dx$$

Solution:

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{8}} \rho^4\,\cos^2\varphi\,\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

6. (10 points) A particle starts at the point $(2, 0)$, moves along the upper semicircle $y = \sqrt{4 - x^2}$, and returns to its starting point along the x-axis. Use Green's Theorem to find the work done on the particle by the force $\mathbf{F}(x, y) = -y^3\,\mathbf{i} + x^3\,\mathbf{j}$.

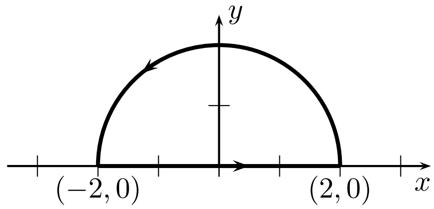

Figure 3: Problem 6

Solution: We want to evaluate the line integral

$$\oint_C -y^3\,dx + x^3\,dy.$$

By Stokes' Theorem, this equals

$$\iint_D \left(\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(-y^3)\right)\,dA = 3\iint_D (x^2 + y^2)\,dA$$

where D is the semicircular region $0 \le y \le \sqrt{4 - x^2}$. Switching to polar coordinates, the last integral equals

$$3\int_0^{\pi}\int_0^2 r^3\,dr\,d\theta = 12\,\pi.$$

7. (7 points)

(a) Let $\mathbf{F}(x, y, z) = y^2\,e^z\,\mathbf{i} + (2xy\,e^z)\,\mathbf{j} + (xy^2\,e^z + 3z^2)\,\mathbf{k}$. Find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

Solution: $f(x, y, z) = x\,y^2\,e^z + z^3$.

(b) Evaluate the line integral

$$\int_C y^2\,e^z\,dx + 2xy\,e^z\,dy + (xy^2\,e^z + 3z^2)\,dz,$$

where C is any curve from $(2, 1, 0)$ to $(1, -1, 1)$.

Solution: The above integral is $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z)$ is the vector field in Part (a). Hence, by the Fundamental Theorem of Line Integrals, it equals

$$x\,y^2\,e^z + z^3\Big|_{(2,1,0)}^{(1,-1,1)} = e - 1.$$

8. (13 points)

(a) Compute the surface integral

$$\iint_S \sqrt{x^2 + y^2 + z^2}\, dS$$

where S is the part of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 1$.

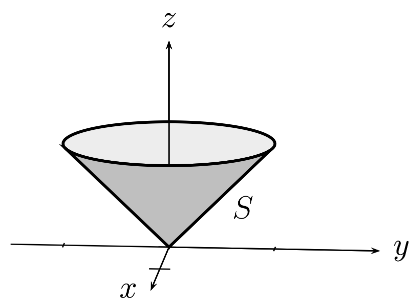

Figure 4: Problem 8a

Solution: The surface projects onto the unit disk $D: \ x^2 + y^2 \le 1$ in the (x, y)–plane. Hence

$$\begin{aligned}
\iint_S \sqrt{x^2 + y^2 + z^2}\, dS &= \iint_D \sqrt{x^2 + y^2} \cdot \left\{ \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \right\} dx\, dy \\
&= \iint_D \sqrt{x^2 + y^2} \cdot \left\{ \sqrt{\left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + 1} \right\} dx\, dy \\
&= \iint_D \sqrt{x^2 + y^2} \cdot \sqrt{2}\, dx\, dy \\
&= \sqrt{2} \int_0^{2\pi} \int_0^1 r^2\, dr\, d\theta \\
&= \frac{2\sqrt{2}\,\pi}{3}.
\end{aligned}$$

(b) Let $\mathbf{F}(x, y, z) = x^2\, \mathbf{i} + yx\, \mathbf{j} + zx\, \mathbf{k}$. Compute the flux

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the part of the plane $2x + y + z = 2$ in the first octant, oriented upward.

Solution: The surface S is the part of the graph $z - 2 - 2x - y$ in the first octant. It projects onto the triangular region D given by $0 \le y \le 2 - 2x,\ 0 \le x \le 1$ in the (x, y)–plane. Hence

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(-P \frac{\partial z}{\partial x} - Q \frac{\partial z}{\partial y} + R \right) dA \\
&= \iint_D \left(-x^2 (-2) - yx(-1) + zx \right) dA \\
&= \iint_D x(2x + y + z)\, dA \\
&= \int_0^1 \int_0^{2-2x} 2x\, dy\, dx \\
&= \int_0^1 2x(2 - 2x)\, dx \\
&= \frac{2}{3}.
\end{aligned}$$

9. (10 points) Let $\mathbf{F}(x, y, z) = (z - y)\,\mathbf{i} + (z + x)\,\mathbf{j} - (x + y)\,\mathbf{k}$. Use Stokes' Theorem to evaluate the flux

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

where S is the part of the paraboloid $z = 9 - x^2 - y^2$ above the (x, y)-plane, with the upward orientation.

Solution 1: The boundary C of S is the circle of radius 3 in the (x, y)–plane, oriented counterclockwise. It is given by the parametric equations

$$x = 3\cos\theta, \qquad y = 3\sin\theta, \qquad z = 0.$$

Hence by Stokes' Theorem,

$$\begin{aligned}
\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\
&= \oint_C (z - y)\, dx + (z + x)\, dy - (x + y)\, dz \\
&= \int_0^{2\pi} \left((-3\sin\theta)(-3\sin\theta) + (3\cos\theta)(3\cos\theta) \right) d\theta \\
&= 9 \int_0^{2\pi} d\theta \\
&= 18\,\pi.
\end{aligned}$$

Let D be the disk $x^2 + y^2 \leq 9$ in the (x, y)–plane. Then since $z = 0$, the second integral on the right above can also be seen to equal $\oint_C -y\, dx + x\, dy = 2 \cdot \operatorname{area}(D) = 18\,\pi$.

Solution 2: S has the same boundary as the flat disk $S_1 :\ z = 0,\ x^2 + y^2 \leq 9$ in the (x, y)–plane, oriented upward. Hence by Stokes' Theorem applied twice,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

But curl $\mathbf{F} = \cdots + 2\,\mathbf{k}$, so

$$
\begin{aligned}
= \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dS \\
&= \iint_{S_1} 2\, dS \\
&= 2 \cdot \text{area}\,(S_1) \\
&= 18\,\pi.
\end{aligned}
$$

10. (10 points) Use the Divergence Theorem to compute the outward flux of

$$\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + z^2\,\mathbf{k}$$

across the boundary of the solid hemisphere $0 \leq z \leq \sqrt{4 - x^2 - y^2}$. **Solution:** Let E be the solid hemisphere and S its boundary. Then by the Divergence Theorem,

$$
\begin{aligned}
\iint_S \mathbf{D} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F}\, dV \\
&= \iiint_E 4z\, dV \\
&= \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} 2rz\, dz\, dr\, d\theta \\
&= 2\pi \int_0^2 r\,(4 - r^2)\, dr \\
&= 8\,\pi.
\end{aligned}
$$

END OF EXAM.