Math 13 | Tufts University | May 9, 2011

Department of Mathematics

Final Exam

No calculators, cell phones, iPods, notes, scratch paper or books are allowed. Except for Problem 1, you must show all your work in your blue book in order to receive a full credit. A correct answer with no work may not necessarily score any points. **Simplify** *your answers as much as possible. Cross out any work you do not want graded. Sign your exam book, indicating that you have neither given nor received help during this exam. Any violations will be reported to the appropriate dean, and will result in an F for the course.*

1. (10 points) **True or False.** Write your answers to this problem on the inside front cover of your blue book. Write **T** if the statement is always true, and write **F** otherwise.

   (a) The lines $x = 2 + t,\ y = -1 - t,\ z = -t$ and $x = 1 - t,\ y = 4 + t,\ z = 1 + t$ are parallel.

   (b) The area of a parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u} \times \mathbf{v}$.

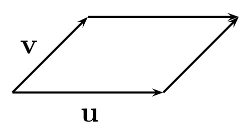

Figure 1: Problem 1b

   (c) If $f_x = x^2 - y$ and $f_y = y^2 - x$, then $(1, 1)$ is a saddle point of $f(x, y)$.

   (d) The line integral $\int_C \nabla f \cdot d\mathbf{r}$ is independent of path.

   (e) $\displaystyle\int_0^2 \int_x^{\sqrt{8-x^2}} y^2\, dy\, dx = \int_0^{\pi/2} \int_0^{\sqrt{8}} r^3 \sin^2\theta\, dr\, d\theta.$

2. (7 points) Find the equation of the tangent plane to the parametric surface

$$x = u \cos v,\ y = u \sin v,\ z = v$$

   at the point $(0, 3, \frac{\pi}{2})$.

3. (8 points) Let $f(x, y) = x^2 e^{-y}$.

   (a) At the point $(2, 0)$, find the direction in which $f(x, y)$ increases most rapidly. Express the direction as a unit vector.

   (b) Find a unit vector $\mathbf{u}$ such that the directional derivative $D_{\mathbf{u}} f(2, 0) = 0$.

The exam continues on the other side.

4. (10 points) Use Lagrange multipliers to find the volume of the largest rectangular box, with edges parallel to the coordinate axes, which can be inscribed in the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$.

5. (15 points)

   (a) The figure below shows the region of integration for the integral

   $$\int_0^1 \int_x^1 \int_0^{1-y^2} f(x, y, z)\, dz\, dy\, dx.$$

   Rewrite this integral as an equivalent iterated integral in the order $dy\, dz\, dx$.

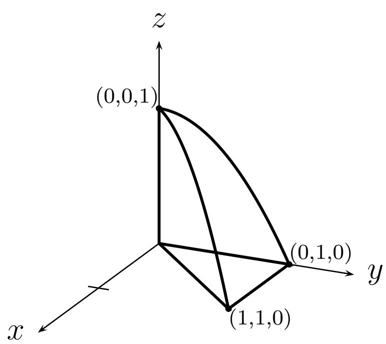

Figure 2: Problem 5a

   (b) Let E be the solid below the sphere $x^2 + y^2 + z^2 = 25$ and above the plane $z = 4$. Express the volume of E as an iterated triple integral in cylindrical coordinates. **Do not evaluate.**

   (c) Convert the triple integral

   $$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2\, dz\, dy\, dx$$

   to an iterated triple in spherical coordinates. **Do not evaluate.**

6. (10 points) A particle starts at the point $(2, 0)$, moves along the upper semicircle $y = \sqrt{4 - x^2}$, and returns to its starting point along the x-axis. Use Green's Theorem to find the work done on the particle by the force $\mathbf{F}(x, y) = -y^3\, \mathbf{i} + x^3\, \mathbf{j}$.

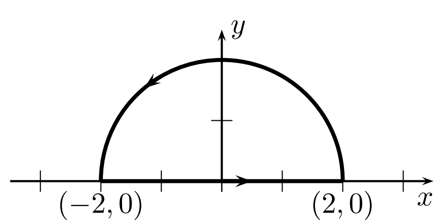

Figure 3: Problem 6

The exam continues on the next page.

7. (7 points)

   (a) Let $\mathbf{F}(x, y, z) = y^2 e^z \,\mathbf{i} + (2xy\, e^z)\,\mathbf{j} + (xy^2 e^z + 3z^2)\,\mathbf{k}$. Find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

   (b) Evaluate the line integral

$$\int_C y^2 e^z \, dx + 2xy\, e^z \, dy + (xy^2 e^z + 3z^2)\, dz,$$

   where C is any curve from $(2, 1, 0)$ to $(1, -1, 1)$.

8. (13 points)

   (a) Compute the surface integral

$$\iint_S \sqrt{x^2 + y^2 + z^2}\, dS$$

   where S is the part of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 1$.

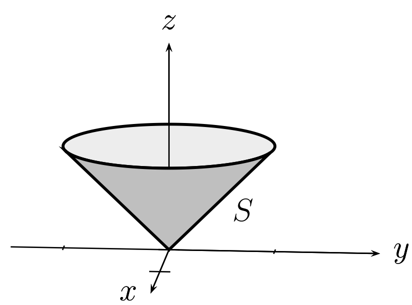

Figure 4: Problem 8a

   (b) Let $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + yx\,\mathbf{j} + zx\,\mathbf{k}$. Compute the flux

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

   where S is the part of the plane $2x + y + z = 2$ in the first octant, oriented upward.

9. (10 points) Let $\mathbf{F}(x, y, z) = (z - y)\,\mathbf{i} + (z + x)\,\mathbf{j} - (x + y)\,\mathbf{k}$. Use Stokes' Theorem to evaluate the flux

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

   where S is the part of the paraboloid $z = 9 - x^2 - y^2$ above the (x, y)-plane, with the upward orientation.

10. (10 points) Use the divergence theorem to compute the outward flux of

$$\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + z^2\,\mathbf{k}$$

   across the boundary of the solid hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$.

END OF EXAM.