Math 13 | Tufts University | April 11, 2011

Department of Mathematics

Exam 2 Solutions

1. (5 points) **Using symmetry, determine if the following integrals are *positive, negative*, or *zero* for the regions shown in Figures 1 and 2. (You don't need to justify your answer.)**

(a) $\displaystyle\iint_R x^2 y \, dA$

(b) $\displaystyle\iint_S \cos x \sin y \, dA$

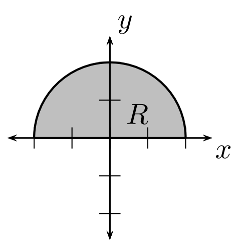


Figure 1: Problem 1a, region R

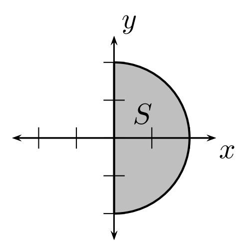


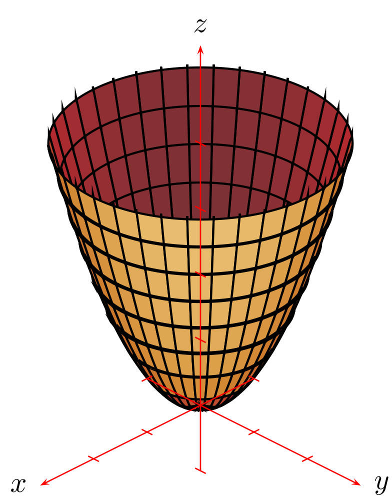
Figure 2: Problem 1b, region S

Solution: Since $x^2 y$ is even in x and the region is contained above the x-axis (hence $y \geq 0$) we have $\boxed{\displaystyle\iint_R x^2 y \, dA > 0}$

Solution: Since $\cos x \sin y$ is odd in y region is symmetric about the x-axis we have $\boxed{\displaystyle\iint_R \cos x \sin y \, dA = 0}$

2. (5 points) **Sketch the surface whose equation in cylindrical coordinates is given by $z = r^2$:**

The graph is a paraboloid opening along the positive z-axis.



3. (5 points) **Let D be the region contained inside the circle of radius $\sqrt{2}$ centered at the origin, above the line $y = x$, and in the first quadrant. (See Figure 3.) Express the integral**

$$\iint_D x^2 \, dA$$

as an iterated integral in *rectangular* coordinates. *Do not evaluate.* (Hint: one order is much simpler.)

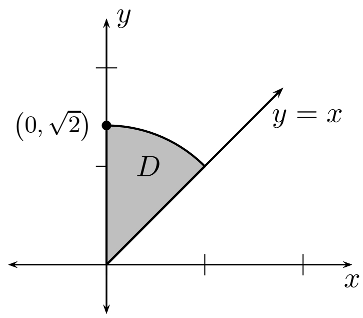


Figure 3: Problem 3

Solution: $\boxed{\int_0^1 \int_x^{\sqrt{2-x^2}} x^2 \, dy \, dx.}$

4. (15 points) **Use the method of Lagrange multipliers to find the maximum value of the function**

$$f(x, y, z) = x^2yz$$

on the surface with equation

$$x^2 + 2y^2 + z^4 = 7.$$

We need to solve the system of equations

$$\begin{aligned} 2xyz &= 2\lambda x \\ x^2z &= 4\lambda y \\ x^2y &= 4\lambda z^3 \\ x^2 + 2y^2 + z^4 &= 7. \end{aligned}$$

Note that if $x = 0, y = 0$ or $z = 0$ then $f(x, y, z) = 0$ which is clearly not maximal. Thus we may freely divide by x, y, or z. Solving the first 3 equations for λ we obtain

$$\lambda = yz = \frac{x^2z}{4y} = \frac{x^2y}{4z^3}$$

hence

$$4y^2 = x^2 \qquad \text{and} \qquad \boxed{x^2 = 4z^4.}$$

We thus have $4y^2 = 4z^4$ or $\boxed{y^2 = z^4.}$ Substituting into the 4th equation we have

$$7 = x^2 + 2y^2 + z^4 = 4z^4 + 2z^4 + z^4$$

and thus $z^4 = 1$ and $\boxed{z = \pm 1.}$ Thus the maximum values occur when

$$(x, y, z) = (\pm 2, \pm 1, \pm 1)$$

and the maximum value of f is

$$\boxed{f(2, 1, 1) = 4.}$$

5. (15 points) **Consider the integral**

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} \left(x^2 + y^2 + z^2\right) \, dz \, dy \, dx.$$

(a) **Rewrite the integral as an iterated integral in *cylindrical coordinates.***

$$\boxed{\int_0^{\pi} \int_0^3 \int_r^{\sqrt{18-r^2}} (r^2 + z^2) r \, dz \, dr \, d\theta.}$$

(b) **Rewrite the integral as an iterated integral in *spherical coordinates.***

$$\boxed{\int_0^{\pi/4} \int_0^{\pi} \int_0^{\sqrt{18}} (\rho^2)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.}$$

6. (10 points) **Figure 4 shows the solid bounded by the surfaces**

$$x = 0, \quad z = 0, \quad y = 2 - z, \quad \textbf{and} \quad y = \sqrt{x}.$$

Set up iterated *triple* integrals that yield the volume of the solid in the following orders. *Do not evaluate!*

(a) $dz \; dy \; dx$:

$$\boxed{\int_0^4 \int_{\sqrt{x}}^2 \int_0^{2-y} 1 \, dz \, dy \, dx.}$$

(b) $dx \; dz \; dy$:

$$\boxed{\int_0^2 \int_0^{2-y} \int_0^{y^2} 1 \, dx \, dz \, dy.}$$

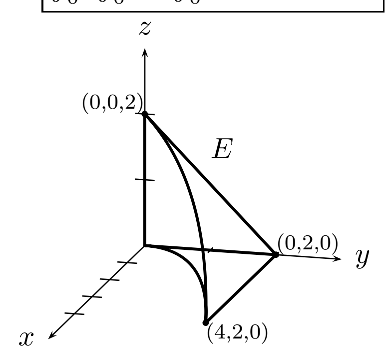


Figure 4: Problem 6

7. (15 points) **Figure 5 shows a solid E. The solid is contained in the first octant, below the cone $z = \sqrt{x^2 + y^2}$, and inside the sphere $x^2 + y^2 + z^2 = 4$. Using spherical coordinates, evaluate the integral**

$$\iiint_E z \, dV.$$

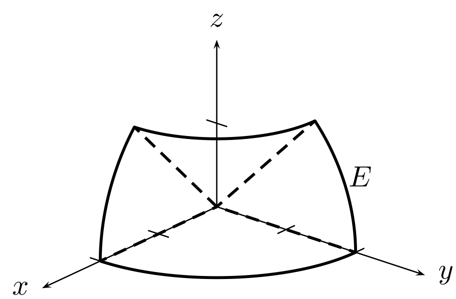


Figure 5: Problem 7

We rewrite the integral in spherical coordinates:

$$\begin{aligned}
\iiint_E z \, dV &= \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^2 (\rho \cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_{\pi/4}^{\pi/2} \cos\phi \sin\phi \, d\phi \int_0^{\pi/2} d\theta \int_0^2 \rho^3 \, d\rho \\
&= \left(\frac{1}{2} \sin^2(\phi) \Big|_{\pi/4}^{\pi/2} \right) \left(\frac{\pi}{2} \right) \left(\frac{\rho^4}{4} \Big|_0^2 \right) \\
&= \left(\frac{1}{4} \right) \left(\frac{\pi}{2} \right) \left(\frac{16}{4} \right) \\
&= \boxed{\frac{\pi}{2}}.
\end{aligned}$$

8. (10 points) **Evaluate the line integral**

$$\int_C x \, ds$$

where C is the curve in the xy-plane parametrized by $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}t^2\,\mathbf{j}$, for $0 \le t \le 2$:
We have $\|\mathbf{r}'(t)\| = \sqrt{1 + t^2}$ hence

$$\begin{aligned}
\int_C x \, ds &= \int_0^2 t\sqrt{1 + t^2} \, dt \\
&= \frac{1}{3}(1 + t^2)^{3/2} \Big|_0^2 \\
&= \boxed{\frac{1}{3}5^{3/2} - \frac{1}{3}}.
\end{aligned}$$

9. (10 points) **Evaluate the line integral**

$$\int_C x^2\ dx + 2\ dy$$

where C is the semicircle of radius 2 shown in Figure 6:

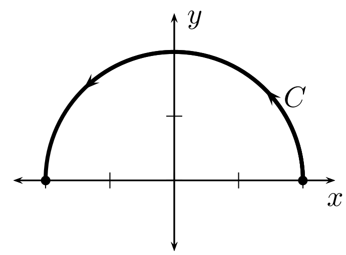


Figure 6: Problem 9

We parametrize the curve by $\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle$ for $0 \le t \le \pi$. We then have $dx = -2\sin t\ dt$ and $dy = 2\cos t\ dt$. Thus

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^\pi (4\cos^2 t)(-2\sin t\ dt) + 2(2\cos t\ dt)\\ &= \int_0^\pi -8\cos^2 t\sin t + 4\cos t\ dt\\ &= \left(\frac{8}{3}\cos^3 t + 4\sin t\right)\Big|_0^\pi\\ &= \left(\frac{-8}{3} - \frac{8}{3}\right) + (4(0) - 4(0))\\ &= \boxed{-\frac{16}{3}.}\end{aligned}$$

10. (10 points) **The vector field $\mathbf{F}(x, y, z) = 2\sin y\,\mathbf{i} + (2x\cos y - z^3 y)\,\mathbf{j} - \frac{3}{2}z^2y^2\,\mathbf{k}$ is conservative.**

(a) **Find a potential function for the vector field $\mathbf{F}(x, y, z)$:**
We have $f_x = 2\sin y$ hence $f(x, y, z) = 2x\sin y + g(y, z)$. We then compare two expressions for f_y:

$$2x\cos y + g_y(y, z) = f_y = 2x\cos y - z^3 y$$

hence we conclude that $g_y(y, z) = -z^3 y$ and $g(y, z) = -\frac{1}{2}z^3y^2 + h(z)$ and

$$f(x, y, z) = 2x\sin y - \frac{1}{2}z^3y^2 + h(z).$$

Now compare two expressions for f_z:

$$-\frac{3}{2}z^2y^2 + h'(z) = f_z = -\frac{3}{2}z^2y^2$$

hence $h'(z) = 0$ and we may take $h(z)$ to be any constant function. Taking $h(z) = 0$ we have that

$$\boxed{f(x, y, z) = 2x\sin y - \frac{1}{2}z^3y^2}$$

is a potential function for $\mathbf{F}(x, y, z)$. In general, potential functions are of the form

$$\boxed{f(x, y, z) = 2x \sin y - \frac{1}{2} z^3 y^2 + K}$$

for any constant K.

(b) **Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve parametrized by**

$$\mathbf{r}(t) = t^2\, \mathbf{i} + t\, \mathbf{j} + \cos t\, \mathbf{k}, \qquad 0 \le t \le \pi :$$

We use the fundamental theorem of line integrals with end points $\mathbf{r}(0) = \langle 0, 0, 1\rangle$ and $\mathbf{r}(\pi) = \langle \pi^2, \pi, -1\rangle$:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= f(\pi^2, \pi, -1) - f(0, 0, 1) \\ &= \left(2(\pi^2) \sin \pi - \frac{1}{2}(-1)^3 \pi^2\right) - \left(2(0) \sin(0) - \frac{1}{2}(1^3)(0^2)\right) \\ &= \boxed{\frac{\pi^2}{2}.} \end{aligned}$$