1. (5 points) Using symmetry, determine if the following integrals are positive, negative, or zero for the regions shown in Figures 1 and 2. (You don’t need to justify your answer.)

(a) \[ \int_R x^2y \, dA \]
(b) \[ \int_S \cos x \sin y \, dA \]

Solution: Since \(x^2y\) is even in \(x\) and the region is contained above the \(x\)-axis (hence \(y \geq 0\)) we have \[ \int_R x^2y \, dA > 0 \]

Solution: Since \(\cos x \sin y\) is odd in \(y\) the region is symmetric about the \(x\)-axis we have \[ \int_R \cos x \sin y \, dA = 0 \]

2. (5 points) Sketch the surface whose equation in cylindrical coordinates is given by \(z = r^2\):

The graph is a paraboloid opening along the positive \(z\)-axis.

\[ z = r^2 \]
3. (5 points) Let \( D \) be the region contained inside the circle of radius \( \sqrt{2} \) centered at the origin, above the line \( y = x \), and in the first quadrant. (See Figure 3.) Express the integral
\[
\int \int_{D} x^2 \, dA
\]
as an iterated integral in rectangular coordinates. Do not evaluate. (Hint: one order is much simpler.)

Solution: \[ \int_0^1 \int_{\sqrt{2}-x^2}^x x^2 \, dy \, dx. \]

4. (15 points) Use the method of Lagrange multipliers to find the maximum value of the function
\[ f(x, y, z) = x^2yz \]
on the surface with equation
\[ x^2 + 2y^2 + z^4 = 7. \]

We need to solve the system of equations
\[
\begin{align*}
2xyz &= 2\lambda x \\
x^2z &= 4\lambda y \\
x^2y &= 4\lambda z^3 \\
x^2 + 2y^2 + z^4 &= 7.
\end{align*}
\]
Note that if \( x = 0, y = 0 \) or \( z = 0 \) then \( f(x, y, z) = 0 \) which is clearly not maximal. Thus we may freely divide by \( x, y, \) or \( z \). Solving the first 3 equations for \( \lambda \) we obtain
\[
\lambda = yz = \frac{x^2z}{4y} = \frac{x^2y}{4z^3}
\]
hence
\[
4y^2 = x^2 \quad \text{and} \quad x^2 = 4z^4.
\]
We thus have \( 4y^2 = 4z^4 \) or \( y^2 = z^4 \). Substituting into the 4th equation we have
\[
7 = x^2 + 2y^2 + z^4 = 4z^4 + 2z^4 + z^4
\]
and thus $z^4 = 1$ and $z = \pm 1$. Thus the maximum values occur when 

$$(x, y, z) = (\pm 2, \pm 1, \pm 1)$$

and the maximum value of $f$ is 

$$f(2, 1, 1) = 4.$$ 

5. (15 points) **Consider the integral**

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dy \, dx.$$

(a) **Rewrite the integral as an iterated integral in cylindrical coordinates.**

$$\int_{0}^{\pi} \int_{0}^{3} \int_{0}^{\sqrt{18-r^2}} (r^2 + z^2) r \, dz \, dr \, d\theta.$$

(b) **Rewrite the integral as an iterated integral in spherical coordinates.**

$$\int_{0}^{\pi/4} \int_{0}^{\pi} \int_{0}^{\sqrt{18}} (\rho^2) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.$$

6. (10 points) **Figure 4 shows the solid bounded by the surfaces**

$$x = 0, \quad z = 0, \quad y = 2 - z, \quad \text{and} \quad y = \sqrt{x}.$$ 

Set up iterated *triple* integrals that yield the volume of the solid in the following orders. *Do not evaluate!*

(a) $dz \, dy \, dx$:

$$\int_{0}^{\sqrt{4-y}} \int_{0}^{2-y} \int_{0}^{4} 1 \, dz \, dy \, dx.$$

(b) $dx \, dz \, dy$:

$$\int_{0}^{2} \int_{0}^{2-y} \int_{0}^{y^2} 1 \, dx \, dz \, dy.$$
7. (15 points) Figure 5 shows a solid $E$. The solid is contained in the first octant, below the cone $z = \sqrt{x^2 + y^2}$, and inside the sphere $x^2 + y^2 + z^2 = 4$. Using spherical coordinates, evaluate the integral

$$\iiint_E z \, dV.$$ 

We rewrite the integral in spherical coordinates:

$$\iiint_E z \, dV = \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^2 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \int_{\pi/4}^{\pi/2} \cos \phi \sin \phi \, d\phi \int_0^{\pi/2} d\theta \int_0^2 \rho^3 \, d\rho$$

$$= \left( \frac{1}{2} \sin^2(\phi) \right)_{\pi/4}^{\pi/2} \left( \frac{\pi}{2} \right) \left( \frac{\rho^4}{4} \right)_{0}^{2}$$

$$= \left( \frac{1}{4} \right) \left( \frac{\pi}{2} \right) \left( \frac{16}{4} \right)$$

$$= \frac{\pi}{2}$$

8. (10 points) Evaluate the line integral

$$\int_C x \, ds$$

where $C$ is the curve in the $xy$-plane parametrized by $r(t) = t \mathbf{i} + \frac{1}{2}t^2 \mathbf{j}$, for $0 \leq t \leq 2$.

We have $\|r'(t)\| = \sqrt{1 + t^2}$ hence

$$\int_C x \, ds = \int_0^2 t \sqrt{1 + t^2} \, dt$$

$$= \frac{1}{3} \left[ (1 + t^2)^{3/2} \right]_0^2$$

$$= \frac{1}{3} \left( 5^{3/2} - 1 \right).$$
9. (10 points) Evaluate the line integral
\[ \int_C x^2 \, dx + 2 \, dy \]
where \( C \) is the semicircle of radius 2 shown in Figure 6:

![Figure 6: Problem 9](image)

We parametrize the curve by \( r(t) = (2 \cos t, 2 \sin t) \) for \( 0 \leq t \leq \pi \). We then have \( dx = -2 \sin t \, dt \) and \( dy = 2 \cos t \, dt \). Thus
\[
\int_C F \cdot dr = \int_0^\pi (4 \cos^2 t)(-2 \sin t \, dt) + 2(2 \cos t \, dt) \\
= \int_0^\pi -8 \cos^2 t \sin t + 4 \cos t \, dt \\
= \left( \frac{8}{3} \cos^3 t + 4 \sin t \right) \bigg|_0^\pi \\
= \left( -\frac{8}{3} - \frac{8}{3} \right) + (4(0) - 4(0)) \\
= \boxed{\frac{16}{3}}
\]

10. (10 points) The vector field \( F(x, y, z) = 2 \sin y \mathbf{i} + (2x \cos y - z^3 y) \mathbf{j} - \frac{3}{2} z^2 y^2 \mathbf{k} \) is conservative.

(a) Find a potential function for the vector field \( F(x, y, z) \):

We have \( f_x = 2 \sin y \) hence \( f(x, y, z) = 2x \sin y + g(y, z) \). We then compare two expressions for \( f_y \):
\[ 2x \cos y + g_y(y, z) = f_y = 2x \cos y - z^3 y \]

hence we conclude that \( g_y(y, z) = -z^3 y \) and \( g(y, z) = -\frac{1}{2} z^3 y^2 + h(z) \) and
\[ f(x, y, z) = 2x \sin y - \frac{1}{2} z^3 y^2 + h(z). \]

Now compare two expressions for \( f_z \):
\[ -\frac{3}{2} z^2 y^2 + h'(z) = f_z = -\frac{3}{2} z^2 y^2 \]

hence \( h'(z) = 0 \) and we may take \( h(z) \) to be any constant function. Taking \( h(z) = 0 \) we have that
\[ f(x, y, z) = 2x \sin y - \frac{1}{2} z^3 y^2 \]
is a potential function for $\mathbf{F}(x, y, z)$. In general, potential functions are of the form

$$f(x, y, z) = 2x \sin y - \frac{1}{2}z^3y^2 + K$$

for any constant $K$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $C$ is the curve parametrized by

$$\mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j} + \cos t \mathbf{k}, \quad 0 \leq t \leq \pi$$

We use the fundamental theorem of line integrals with end points $\mathbf{r}(0) = \langle 0, 0, 1 \rangle$ and $\mathbf{r}(\pi) = \langle \pi^2, \pi, -1 \rangle$:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\pi^2, \pi, -1) - f(0, 0, 1)$$

$$= \left( 2(\pi^2) \sin \pi - \frac{1}{2}(-1)^3\pi^2 \right) - \left( 2(0) \sin(0) - \frac{1}{2}(1^3)(0^2) \right)$$

$$= \frac{\pi^2}{2}.$$