Tufts University

Math 13	**Department of Mathematics**	**May 10, 2010**
	Final Exam	**8:30 to 10:30 am**

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (20 points) Compute the following.

   (a) The area of the parallelogram formed by the vectors $\mathbf{i}+\mathbf{j}$ and $-\mathbf{i}+\mathbf{j}+\mathbf{k}$.
   Solution: Using the cross-product area formula, we have

$$\text{Area} = \|(\mathbf{i}+\mathbf{j}) \times (-\mathbf{i}+\mathbf{j}+\mathbf{k})\|.$$

   So, we first compute the cross product,

$$(\mathbf{i}+\mathbf{j}) \times (-\mathbf{i}+\mathbf{j}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix} = \mathbf{i}-\mathbf{j}+2\mathbf{k}.$$

   This gives Area $= \sqrt{1+1+4} = \sqrt{6}$.

   (b) An equation for the plane containing the point $(0, 4, 2)$ that is parallel to the plane $x - 5y + 2z = 2159$.
   Solution: The formula for a plane through a point is $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{x}_0$. Recognizing $\mathbf{n} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$, this gives us

$$x - 5y + 2z = (\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}) \cdot (0\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = -16$$

   .

   (c) $\mathbf{a} \cdot \mathbf{b}$ where $|\mathbf{a}| = 3$, $|\mathbf{b}| = 3$ and the vectors $\mathbf{a}$ and $\mathbf{b}$ form an angle of $\frac{3\pi}{4}$.
   Solution: From the formula,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = 3 \cdot 3 \cos(\frac{3\pi}{4}) = -\frac{9\sqrt{2}}{2}.$$

   (d) $\dfrac{dz}{dx}$ at the point $(1, 1, 0)$ where z is defined implicitly via the equation

$$\ln(xy + 4z) = e^{-z} - 1.$$

   Solution: Implicitly differentiating with respect to x, we obtain

$$\frac{y + 4\frac{\partial z}{\partial x}}{xy + 4z} = -\frac{\partial z}{\partial x} e^{-z}.$$

   Substituting in the point values, we obtain

$$\frac{1 + 4\frac{\partial z}{\partial x}}{1} = -\frac{\partial z}{\partial x} 1.$$

   Hence,

$$\frac{\partial z}{\partial x} = \frac{-1}{5}.$$

(e) The directional derivative of $f(x,y) = \sin(xy)$ at $(\pi, 1)$, in the direction of $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$.
Solution: Let $\mathbf{v} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. We have

$$\nabla f(x,y) = y\cos(xy)\mathbf{i} + x\cos(xy)\mathbf{j}$$

hence

$$\begin{aligned} D_{\mathbf{v}} f(\pi, 1) &= (\nabla f(\pi, 1)) \cdot \mathbf{v} \\ &= (-1\mathbf{i} - \pi\mathbf{j}) \cdot (\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}) \\ &= \frac{-1-\pi}{\sqrt{2}} \end{aligned}$$

2. (10 points) Let $f(x,y) = x^3 + y^3$ be defined on the xy-region $D = \{(x,y) \mid x^2 + y^2 \leq 4\}$. Find the absolute maximum and absolute minimum values of f on D and the coordinates where they occur.
Solution: First, we find the critical points of f:

$$f_x = 3x^2 = 0 \text{ and } f_y = 3y^2 = 0 \text{ yield unique CP: } (x,y) = (0,0),$$

where $f(0,0) = 0$. (Note: $(0,0) \in D$)

Next, we find the extreme values of f on the boundary of D. We do this using either Lagrange Multipliers or Polar Coordinates, as illustrated below.

Method I: Using Lagrange Multipliers.

To find the extreme values of $f(x,y) = x^3 + y^3$ subject to the constraint $x^2 + y^2 = 4$ - the equation of the circle that bounds D - we must solve the system:

$$\begin{cases} 3x^2 = \lambda(2x) \\ 3y^2 = \lambda(2y) \\ x^2 + y^2 = 4 \end{cases}$$

Case 1: $x \neq 0$, $y = 0$.

From the last equation in the system, we have right away that $x^2 = 4$ or $x = \pm 2$. These values correspond to the xy-locations $(-2,0)$ and $(2,0)$, so that $f(-2,0) = -8$ and $f(2,0) = 8$.

Case 2: $x = 0$, $y \neq 0$.

Similar reasoning gives xy-locations $(0,-2)$ and $(0,2)$, so that $f(0,-2) = -8$ and $f(0,2) = 8$.

Case 3: $x \neq 0$, $y \neq 0$.

Dividing the first two equations in the system by x and y, respectively, gives $x = y = \frac{2\lambda}{3}$. Replacing these in the last equation of the system will allow us to solve for λ:

$$\frac{4\lambda^2}{9} + \frac{4\lambda^2}{9} = 4 \text{ or } \frac{2\lambda^2}{9} = 1 \text{ or } \lambda = \pm\frac{3\sqrt{2}}{2}.$$

Thus, if $\lambda = -\frac{3\sqrt{2}}{2}$, then $(x, y) = (-\sqrt{2}, -\sqrt{2})$ and, consequently, $f(-\sqrt{2}, -\sqrt{2}) = -4\sqrt{2}$.

If, on the other hand, $\lambda = \frac{3\sqrt{2}}{2}$, then $(x, y) = (\sqrt{2}, \sqrt{2})$ and $f(\sqrt{2}, \sqrt{2}) = 4\sqrt{2}$.

Case 4: $x = 0$ and $y = 0$.
This case is impossible, as it would lead to an immediate contradiction of the constraint equation: $0 = 4$.

Comparing all of the f-values above, including that of the critical point, we see that $f_{min} = -8$ at either $(-2, 0)$ or $(0, -2)$, and that $f_{max} = 8$ at either $(2, 0)$ or $(0, 2)$.

Method II: Using Polar Coordinates.
On the circle $x^2 + y^2 = 4$, $(x, y) = (2\cos\theta, 2\sin\theta)$, so that

$$f(x, y) = x^3 + y^3 = 8\cos^3\theta + 8\sin^3\theta \equiv g(\theta),$$

where g, as a function of θ needs now be maximized/minimized on the interval $[0, 2\pi]$. Using what we learned in Calculus I for g, we have:

$$g'(\theta) = 24\cos^2\theta(-\sin\theta) + 24\sin^2\theta(\cos\theta) = 24\sin\theta\cos\theta(\sin\theta - \cos\theta).$$

Thus, $g'(\theta) = 0$ for $\theta \in \{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, 2\pi\}$.
Exhausting all of these θ-values, one at a time, we have:

$g(0) = f(0, 2) = 8$

$g(\frac{\pi}{4}) = f(\sqrt{2}, \sqrt{2}) = 4\sqrt{2}$

$g(\frac{\pi}{2}) = f(2, 0) = 8$

$g(\pi) = f(-2, 0) = -8$

$g(\frac{5\pi}{4}) = f(-\sqrt{2}, -\sqrt{2}) = -4\sqrt{2}$

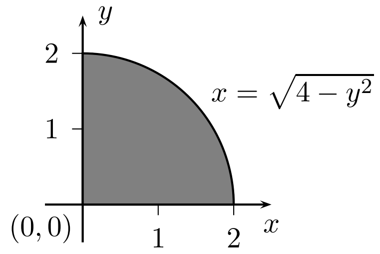
$g(\frac{3\pi}{2}) = f(0, -2) = -8$

$g(2\pi) = f(0, 2) = 8$

Collecting the max/min values from this list and comparing them to the value of f at the critical point, the answer will be identical to that in **Method I**.

3. (10 points) Consider the integral $\int_0^2 \int_0^{\sqrt{4-y^2}} xy\, dx\, dy$.

(a) Sketch the region of integration.
Solution:



(b) Evaluate the integral by switching to polar coordinates.
Solution: Converting to polar coordinates, we have

$$\int_0^2 \int_0^{\sqrt{4-y^2}} xy\, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^2 (r\cos\theta)(r\sin\theta) r\, dr\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \left.\frac{r^4}{4}\right|_0^2 d\theta = 4\left(\frac{1}{2}\sin^2\theta\Big|_0^{\frac{\pi}{2}}\right) = \frac{4}{2} = 2.$$

4. (10 points) Consider the solid volume that lies inside both the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 1$.

(a) Give a **triple** integral in **rectangular** coordinates for the volume of this solid. **Do NOT evaluate this integral!**
Solution: We can bound the region inside of the cylinder by taking $-1 \le x \le 1$ and, for any fixed x, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$. Then, for any fixed x and y, the points within the sphere satisfy $-\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2}$. So, the integral is:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} 1\, dz\, dy\, dx.$$

(b) Change your integral from part (a) to **cylindrical** coordinates, and **compute** the volume.
Solution: In cylindrical coordinates, take $x = r\cos(\theta)$ and $y = r\sin(\theta)$, and $z = z$. This changes the region in the xy-plane to be $\{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$, while the bounds on z become $-\sqrt{4-r^2} \le z \le \sqrt{4-r^2}$. So, the integral is:

$$\int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^1 2r\sqrt{4-r^2}\, dr\, d\theta$$
$$= \int_0^{2\pi} -\frac{2}{3}\left(4-r^2\right)^{\frac{3}{2}}\Big|_0^1 d\theta \text{ (making a } u\text{-substitution)}$$
$$= \int_0^{2\pi} \left(-\frac{2}{3}(3)^{\frac{3}{2}} + \frac{2}{3}(4)^{\frac{3}{2}}\right) d\theta$$
$$= \left(\frac{16}{3} - 2\sqrt{3}\right) 2\pi.$$

5. (10 points) Evaluate the surface integral, $\iint_S xz dS$, where S is the surface with parametric equations $x = u$, $y = u\sin v$, $z = u\cos v$, for $0 \le u \le 2$, $0 \le v \le \frac{\pi}{2}$.
Solution: Writing the parameterized surface as $\mathbf{r}(u, v) = u\mathbf{i} + u\sin v\mathbf{j} + u\cos v\mathbf{k}$, we first compute

$$\begin{aligned} \mathbf{r}_u &= 1\mathbf{i} + \sin v\mathbf{j} + \cos v\mathbf{k} \\ \mathbf{r}_v &= 0\mathbf{i} + u\cos v\mathbf{j} - u\sin v\mathbf{k} \end{aligned}$$

and, then, their cross product

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \sin v & \cos v \\ 0 & u\cos v & -u\sin v \end{vmatrix} = -u\mathbf{i} + u\sin v\mathbf{j} + u\cos v\mathbf{k}.$$

Thus, the "bonus term" for the surface integral is

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \left(u^2 + u^2\sin^2 v + u^2\cos^2 v\right)^{\frac{1}{2}} = \sqrt{2}u,$$

since u is never negative. So, putting everything together, we get

$$\begin{aligned} \iint_S xz dS &= \int_0^{\frac{\pi}{2}} \int_0^2 (u)(u\cos v)(\sqrt{2}u)\, du\, dv \\ &= \int_0^{\frac{\pi}{2}} \int_0^2 \sqrt{2}u^3 \cos v\, du\, dv = \int_0^{\frac{\pi}{2}} \sqrt{2}\, \frac{u^4}{4}\bigg|_0^2 \cos v\, dv \\ &= 4\sqrt{2} \int_0^{\frac{\pi}{2}} \cos v\, dv = 4\sqrt{2}\sin v\bigg|_0^{\frac{\pi}{2}} = 4\sqrt{2}. \end{aligned}$$

6. (10 points) Let $\mathbf{F} = \langle e^y + \sin z,\ xe^y,\ x\cos z\rangle$.

(a) Show that $\mathbf{F}$ is conservative.
Solution: A three-dimensional vector field, $\mathbf{F}$, is conservative if $\nabla \times \mathbf{F} = 0$. So, we compute

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y + \sin z & xe^y & x\cos z \end{vmatrix} = 0\mathbf{i} + (-\cos z + \cos z)\mathbf{j} + (e^y - e^y)\mathbf{k} \\ &= \mathbf{0}\,. \end{aligned}$$

Thus, $\mathbf{F}$ is conservative.

(b) Find a potential function f such that $\nabla f = \mathbf{F}$.
Solution: We know that $f_x = e^y + \sin z$, so integrating with respect to x gives

$$f = xe^y + x\sin z + g(y, z)\,.$$

However, taking the derivatives of f with respect to y and z, we see that g must be a constant, i.e. $f(x, y, z) = xe^y + x\sin z + C$.

(c) **Without** explicitly computing an integral, find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve parameterized by $\mathbf{r}(t) = \langle t, t^2, t+1 \rangle$, $0 \le t \le 1$.

Solution: We have $\mathbf{r}(0) = (0, 0, 1)$ and $\mathbf{r}(1) = (1, 1, 2)$. Thus, by the fundamental theorem for line integrals,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1, 2) - f(0, 0, 1) = e + \sin(2)\,.$$

7. (10 points) Let $\mathbf{F} = \langle e^{x^2 \sin x} + y^2,\ x + \ln(\cos y) \rangle$ and let C be the boundary curve of the unit circle $x^2 + y^2 = 1$, oriented positively. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution: We can write $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, where $P(x, y) = e^{x^2 \sin x} + y^2$ and $Q(x, y) = x + \ln(\cos y)$. Let D denote the unit disk $x^2 + y^2 \le 1$ in $\mathbb{R}^2$. Then, by Green's Theorem,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D (\partial Q/\partial x - \partial P/\partial y) dA \\
&= \iint_D (1 - 2y) dA \\
&= \int_0^{2\pi} \int_0^1 (1 - 2r \sin\theta) r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 (r - 2r^2 \sin\theta) \, dr \, d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{2} r^2 - \frac{2}{3} r^3 \sin\theta \right]_{r=0}^{1} d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{2} - \frac{2}{3} \sin\theta \right) d\theta \\
&= \left[\frac{1}{2}\theta + \frac{2}{3}\cos\theta \right]_{\theta=0}^{2\pi} \\
&= \pi\,.
\end{aligned}$$

8. (10 points) Let C be a simple closed curve contained in the plane $x + 2y + 2z = 5$ that encloses a region of area 3 and is oriented counterclockwise when viewed from above. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = z\mathbf{i} + x\mathbf{j} - y\mathbf{k}$. (*Hint*: First explain how you can use Stokes' Theorem to compute this integral.)

Solution: Stokes' Theorem allows us to rewrite an integral over a simple closed curve, C, in terms of an integral over an oriented surface, S, whose positively oriented boundary is C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

In this case, since we know that C lies in the plane $x + 2y + 2z = 5$, we can take the oriented surface, S, to be the plane with the upward normal vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Normalizing the unit vector, we find

$$\mathbf{n} = \frac{1}{\|\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\|}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \frac{1}{3}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

So,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \iint_S (\nabla\times\mathbf{F})\cdot\left(\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right)dS.$$

Finally, we compute $\nabla\times\mathbf{F}$:

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & -y \end{vmatrix} = -\mathbf{i}+\mathbf{j}+\mathbf{k}.$$

This gives

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S (\nabla\times\mathbf{F})\cdot\left(\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right)dS = \iint_S (-\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\left(\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right)dS \\ &= \iint_S 1\,dS.\end{aligned}$$

However, $\iint_S 1\,dS$ is just the surface area of S, which we know is 3. So,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 3.$$

9. (10 points) Let S be the surface of the solid bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $z=4$, oriented outwards. Compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$ for $\mathbf{F} = (xy^2+\cos(z^3))\mathbf{i} + (yx^2+x^4)\mathbf{j} + z^2\mathbf{k}$.
Solution: Since we are integrating over the boundary of a closed solid, E, we can use the divergence theorem:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \nabla\cdot\mathbf{F}dV.$$

Computing

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(xy^2+\cos(z^3)) + \frac{\partial}{\partial y}(yx^2+x^4) + \frac{\partial}{\partial z}(z^2) = y^2+x^2+2z.$$

The cylinder is easily written in cylindrical coordinates:

$$\{(r,\theta,z) \mid 0\le r\le 2,\ 0\le\theta\le 2\pi,\ 0\le z\le 4\}.$$

So,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \nabla\cdot\mathbf{F}dV \\ &= \int_0^{2\pi}\int_0^2\int_0^4 (r^2+2z)r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 \left(r^3z+rz^2\right)\Big|_0^4 dr\,d\theta = \int_0^{2\pi}\int_0^2 4r^3+16r\,dr\,d\theta \\ &= \int_0^{2\pi}\left(r^4+8r^2\right)\Big|_0^2 d\theta = 48\int_0^{2\pi} d\theta = 96\pi.\end{aligned}$$

End of Exam