Tufts University

Math 13	**Department of Mathematics**	**April 5, 2010**
	Exam 2	**12:00 noon to 1:20 pm**

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (12 points)

   (a) The volume of the ellipsoid
   $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$
   is given by the equation
   $$V(a,b,c)=\frac{4}{3}\pi abc.$$
   If a increases at a rate of 2 units per second, b decreases at a rate of 1 unit per second, and c is held constant, at what rate is the volume of the ellipsoid changing when $a=5, b=5$ and $c=3$.
   Solution: We have $V=\frac{4}{3}\pi abc$, from which we have
   $$\frac{dV}{dt}=\frac{4}{3}\pi bc\frac{da}{dt}+\frac{4}{3}\pi ac\frac{db}{dt}+\frac{4}{3}\pi ab\frac{dc}{dt}.$$
   Substituting, we have
   $$\frac{dV}{dt}=\frac{4}{3}\pi\big(5\cdot 3\cdot 2+5\cdot 3\cdot(-1)+5\cdot 5\cdot 0\big)=20\pi$$

   (b) Let $f(x,y,z)=\frac{x^2}{25}+\frac{y^2}{25}+\frac{z^2}{9}$. Find an equation for the tangent plane to the level surface $f(x,y,z)=1$ at $(x_0,y_0,z_0)=(3,4,0)$.
   Solution: We know that the tangent plane to $f(x,y,z)$ at (x_0,y_0,z_0) satisfies
   $$\nabla f(x_0,y_0,z_0)\cdot((x-x_0)\mathbf{i}+(y-y_0)\mathbf{j}+(z-z_0)\mathbf{k})=0.$$
   Computing, we have
   $$\begin{aligned}\nabla f(x_0,y_0,z_0)&=\frac{2x_0}{25}\mathbf{i}+\frac{2y_0}{25}\mathbf{j}+\frac{2z_0}{25}\mathbf{k}\\&=\frac{6}{25}\mathbf{i}+\frac{8}{25}\mathbf{j}+\frac{0}{25}\mathbf{k}.\end{aligned}$$
   This gives
   $$\left(\frac{6}{25}\mathbf{i}+\frac{8}{25}\mathbf{j}+\frac{0}{25}\mathbf{k}\right)\cdot((x-3)\mathbf{i}+(y-4)\mathbf{j}+(z-0)\mathbf{k})=0,$$
   or $3x+4y=25$.

2. (12 points) Find and classify (as local minima, local maxima, or saddle points) all critical points of $f(x,y) = e^{-x}(x^2 - y^2)$.
Solution: Computing the first partial derivatives we get

$$\frac{\partial f}{\partial x} = -e^{-x}(x^2 - y^2) + e^{-x}(2x) \qquad \frac{\partial f}{\partial y} = e^{-x}(-2y).$$

Critical points are points where these partial derivatives are both zero. Since e^{-x} is never zero, $\frac{\partial f}{\partial y} = 0$ means that $y = 0$. Substituting this into the equation for $\frac{\partial f}{\partial x} = 0$ gives $2x - x^2 = 0$, which has solutions $x = 0, 2$.

So, there are two critical points, $(0,0)$ and $(2,0)$.

Now, calculate the second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = -e^{-x}(-x^2 + y^2 + 2x) + e^{-x}(2 - 2x) \qquad \frac{\partial^2 f}{\partial x \partial y} = e^{-x}(2y) \qquad \frac{\partial^2 f}{\partial y^2} = e^{-x}(-2).$$

At $(0,0)$, $D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -4$. Since this is negative, $(0,0)$ is a saddle point.

At $(2,0)$, $D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 4e^{-4}$. Since this is positive, but $\frac{\partial^2 f}{\partial x^2} = -2e^{-2}$ is negative, $(2,0)$ is a local maxima.

3. (12 points) Use the Method of Lagrange Multipliers to find three positive numbers whose sum is 300 and whose product is maximal. *No credit will be given for a solution that does not use Lagrange Multipliers!*
Solution: Let $f(x,y,z) = xyz$ and $g(x,y,z) = x + y + z$. Then the problem asks us to maximize $f(x,y,z)$ subject to the constraint that $g(x,y,z) = 300$. This is the sort of problem that we solve using Lagrange multipliers.

Compute $\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = 1\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}$. Setting

$$\nabla f = \lambda \nabla g$$
$$g(x,y,z) = 300$$

gives us

$$yz = \lambda \tag{1}$$
$$xz = \lambda \tag{2}$$
$$xy = \lambda \tag{3}$$
$$x + y + z = 300. \tag{4}$$

From (1) and (2), we have that $yz = xz$, so either $z = 0$ or $x = y$. We can't have $z = 0$, since this won't give a maximum of $f(x,y,z)$. Similarly, from (1) and (3), we have that $yz = xy$, so either $y = 0$ or $x = z$. Again, we can't have $y = 0$, since this won't give a maximum of $f(x,y,z)$.

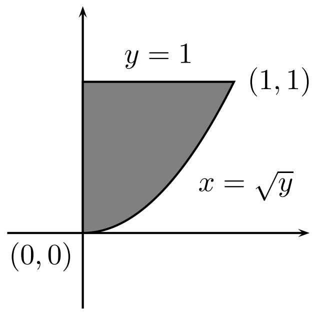
Taking $x = y = z$ in (4) gives $3x = 300$, or $x = 100$. Thus, the maximum occurs when $x = y = z = 100$, and is $f(100,100,100) = 1000000$.

4. (12 points)*Set up and evaluate* a **double** integral for the volume of the region in $\mathbb{R}^3$ above the rectangle $\left[0, \frac{\pi}{2}\right] \times [0, 2]$ and below the graph $z = \dfrac{y}{2} \cos x \sin x$.
Solution: We have

$$\begin{aligned} \text{Volume} &= \int_0^{\frac{\pi}{2}} \int_0^2 \frac{y}{2} \cos x \sin x \, dy \, dx \\ &= \int_0^{\frac{\pi}{2}} \left[\frac{1}{4} y^2\right]_{y=0}^2 \cos x \sin x \, dy \, dx \\ &= \int_0^{\frac{\pi}{2}} \cos x \sin x \, dy \, dx \\ &= \left[\frac{1}{2} \sin^2 x\right]_{x=0}^{\frac{\pi}{2}} \\ &= \frac{1}{2} . \end{aligned}$$

5. (12 points) Consider the integral $\displaystyle\int_0^1 \int_{x^2}^1 x^3 \sin(y^3) \, dy \, dx$.

   (a) Sketch the region of integration.
   Solution:



   (b) Evaluate the integral by reversing the order of integration.
   Solution: Reversing the order, we get:

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3) \, dx \, dy &= \frac{1}{4} \int_0^1 \left[x^4\right]_{x=0}^{\sqrt{y}} \sin(y^3) \, dy \\ &= \frac{1}{4} \int_0^1 y^2 \sin(y^3) \, dy \\ &= -\frac{1}{12} \left[\cos(y^3)\right]_{y=0}^1 \\ &= -\frac{1}{12} (\cos 1 - 1) . \end{aligned}$$

6. (12 points) Let E be the wedge in the first octant bounded by the cylinder $x^2 + y^2 = 1$ and by the planes $x = z$, $y = 0$ and $z = 0$.

   (a) *Set up* a **triple** integral in rectangular coordinates, in the order $dz\, dy\, dx$, for the volume of E. **Do *not* evaluate this integral.**
   Solution: Since E is in the first octant, we have $x \geq 0$, $y \geq 0$, $z \geq 0$. The outermost integrals are over x and y, which are further constrained by the curve, $x^2 + y^2 = 1$. This gives $0 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1 - x^2}$. The z-coordinate is bounded by the plane $x = z$, so $0 \leq z \leq x$. This gives us

$$V(E) = \iiint_E 1\, dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^x 1\, dz\, dy\, dx.$$

   (b) Convert your integral from part (a) to cylindrical coordinates and **compute the volume** of E.
   Solution: The region of the disk $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$ can be written in polar coordinates as $\{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}$. In the limits on z, we have $0 \leq z \leq x = r\cos(\theta)$, giving

$$\begin{aligned} V(E) &= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{r\cos\theta} r\, dz\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cos\theta\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1^3}{3} - \frac{0^3}{3}\right) \cos\theta\, d\theta = \frac{1}{3}\left(\sin\theta\Big|_0^{\frac{\pi}{2}}\right) = \frac{1}{3} \end{aligned}$$

7. (12 points) Let E be the solid inside the sphere $x^2 + y^2 + z^2 = 1$ and below the cone $z = \sqrt{x^2 + y^2}$. Use spherical coordinates to **compute the volume** of E.
Solution: In spherical coordinates, inside the sphere corresponds to $0 \leq \rho \leq 1$. Since both the cone and sphere are symmetric about the z axis, we have no restriction on θ: $0 \leq \theta \leq 2\pi$. Finally, substituting $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$ into $z = \sqrt{x^2 + y^2}$, we get $\cos(\phi) = \sin(\phi)$, which corresponds to $\phi = \dfrac{\pi}{4}$. Below this cone gives larger values of ϕ: $\dfrac{\pi}{4} \leq \phi \leq \pi$. Thus,

$$\begin{aligned} V(E) &= \iiint_E 1\, dV = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\pi} \int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\pi} \left(\frac{1^3}{3} - \frac{0^3}{3}\right) \sin\phi\, d\phi\, d\theta = \frac{1}{3} \int_0^{2\pi} \left(\cos\phi\Big|_{\frac{\pi}{4}}^{\pi}\right) d\theta \\ &= \frac{1}{3}\left(1 + \frac{\sqrt{2}}{2}\right) \int_0^{2\pi} d\theta = \frac{2\pi}{3}\left(1 + \frac{\sqrt{2}}{2}\right) \end{aligned}$$

8. (16 points) *Evaluate* the following line integrals

(a)

$$\int_C x^3\ dx + xy\ dy$$

where C is the top half ($y \geq 0$) of the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(-2,0)$.
Solution: We parameterize the curve

$$\mathbf{r}(t) = < 2\cos t, 2\sin t > \quad 0 \leq t \leq \pi.$$

Substituting we have

$$\begin{aligned}\int_C x^3\ dx + xy\ dy &= \int_0^\pi 8\cos^3 t(-2\sin t) + (2\cos t)(2\sin t)(2\cos t)\ dt \\ &= \int_0^\pi -16\cos^3 t\sin t + 8\cos^2 t\sin t dt \\ &= \left(4\cos^4 t - \frac{8}{3}\cos^3 t\right)\Big|_0^\pi \\ &= 4 + \frac{8}{3} - (4 - \frac{8}{3}) \\ &= \frac{16}{3}\end{aligned}$$

(b)

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

where $\mathbf{F}(x,y,z) = y\mathbf{i} + (x+z)\mathbf{j} + x^2y\mathbf{k}$ and C is the curve parameterized by $\mathbf{r}(t) = \langle 2t, t^3, t^4\rangle$ for $0 \leq t \leq 1$.
Solution: We substitute

$$\begin{aligned}\int_C \mathbf{F}\cdot\ d\mathbf{r} &= \int_0^1 < t^3, 2t + t^4, 4t^5 > \cdot < 2, 3t^2, 4t^3 >\ dt \\ &= \int_0^1 2t^3 + 6t^3 + 3t^6 + 16t^8\ dt \\ &= (\frac{2}{4}t^4 + \frac{6}{4}t^4 + \frac{3}{7}t^7 + \frac{16}{9}t^9)\Big|_0^1 \\ &= \frac{1}{2} + \frac{3}{2} + \frac{3}{7} + \frac{16}{9} \\ &= \frac{265}{63}\end{aligned}$$

End of Exam