Math 13 | **Tufts University Department of Mathematics Exam 1** | **February 22, 2010 12:00 noon to 1:20 pm**

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (10 points) **True or False - No Partial Credit**: On the first page of your blue book, answer the following questions as **True** or **False**.

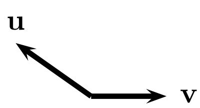


Figure 1: Problem 1

(a) For $\mathbf{u}$ and $\mathbf{v}$ as shown in Figure 1, the vector $\mathbf{u} \times \mathbf{v}$ points into the page.
Solution: True, by the right-hand rule

(b) For $\mathbf{u}$ and $\mathbf{v}$ as shown in Figure 1, $\mathbf{u} \cdot \mathbf{v} > 0$.
Solution: False, since $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos(\theta)$ and $\frac{\pi}{2} < \theta < \pi$.

(c) $\mathbf{u} \cdot \mathbf{w} = 0$ implies that $\mathbf{u}$ and $\mathbf{w}$ are orthogonal.
Solution: True, by definition

(d) The vectors $\langle -4, 6, -10\rangle$ and $\langle 6, -9, 15\rangle$ are parallel.
Solution: True, since $\langle -4, 6, -10\rangle = \frac{-2}{3}\langle 6, -9, 15\rangle$

(e) If $\mathbf{a}$ and $\mathbf{b}$ are parallel then $|\mathbf{a} \times \mathbf{b}| = 0$.
Solution: True, since $\mathbf{a}$ and $\mathbf{b}$ are parallel means that $\mathbf{b} = k\mathbf{a}$ and $\mathbf{a} \times \mathbf{a} = \mathbf{0}$

2. (12 points)

(a) If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ find $\mathbf{a} \times \mathbf{b}$.
Solution:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}\mathbf{k} = -7\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}.$$

(b) Find the scalar projection of $\mathbf{v}$ onto $\mathbf{u}$, $\text{comp}_{\mathbf{u}}\mathbf{v}$, for $\mathbf{u} = \langle 2, 1\rangle$ and $\mathbf{v} = \langle 1, 3\rangle$
Solution:

$$\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} = \frac{2 \cdot 1 + 1 \cdot 3}{\sqrt{2^2 + 1^2}} = \frac{5}{\sqrt{5}} = \sqrt{5}.$$

(c) Find the angle between the vectors $\langle 1, 2, -1\rangle$ and $\langle 2, 1, 1\rangle$.
Solution:

$$\cos\theta = \frac{\langle 1, 2, -1\rangle \cdot \langle 2, 1, 1\rangle}{|\langle 1, 2, -1\rangle|\;|\langle 2, 1, 1\rangle|} = \frac{3}{6} = \frac{1}{2},$$

giving $\theta = \frac{\pi}{3}$.

3. (12 points) Consider the plane $\mathcal{P}$ given by $4x - 2y + 3z = 5$

(a) For what value(s) of b is the line with parametric equations $x(t) = 1 - 2t$, $y(t) = 4 - t$ and $z(t) = 5 + bt$ perpendicular to the normal vector of $\mathcal{P}$?
Solution: The direction vector of this line, $\langle -2, -1, b\rangle$, must be orthogonal to the normal vector of the plane $\mathcal{P}$, $\langle 4, -2, 3\rangle$. Hence, their dot product must be *zero*:

$$\langle -2, -1, b\rangle \cdot \langle 4, -2, 3\rangle = 0$$

Thus,

$$-8 + 2 + 3b = 0 \quad \text{so that} \quad b = 2.$$

(b) At what point does the line with parametric equations $x(t) = 2t$, $y(t) = -1 + t$, and $z(t) = 1 - t$ intersect the plane $\mathcal{P}$?
Solution: At the point of intersection in question, the parametric equations of the line must obey the equation of the plane:

$$4(2t) - 2(-1 + t) + 3(1 - t) = 5 \quad \text{so that} \quad 3t = 0 \quad \text{for which } t = 0.$$

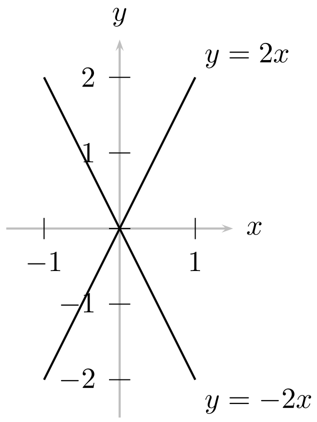
The point of intersection is therefore $(0, -1, 1)$.

(c) Find an equation for the plane that includes the point $(2, -3, 4)$ and is parallel to $\mathcal{P}$.
Solution: Since the two planes are parallel, the normal to $\mathcal{P}$ serves as the normal to the plane in question.

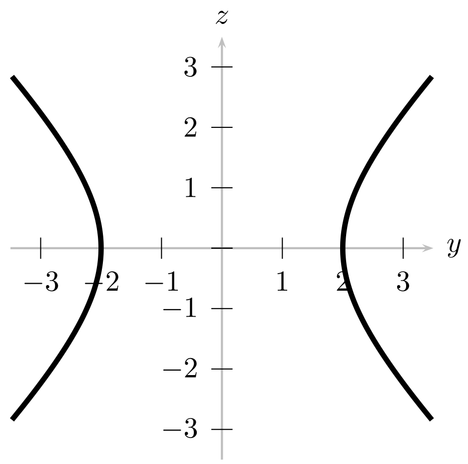
$$4(x - 2) - 2(y + 3) + 3(z - 4) = 0 \quad \text{or} \quad 4x - 2y + 3z - 26 = 0$$

4. (10 points) Consider the surface $\mathcal{S} : y^2 = 4x^2 + z^2$.

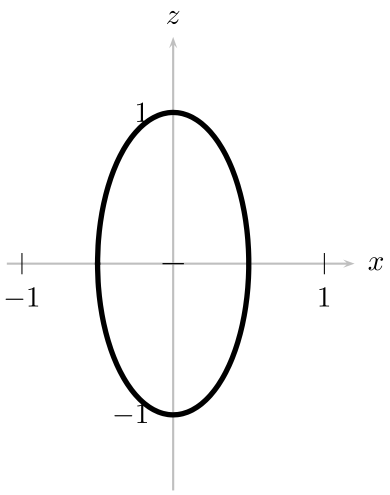
(a) Draw a 2-dimensional sketch that shows the trace of $\mathcal{S}$ in the plane $z = 0$.
Solution: Taking $z = 0$ gives $y^2 = 4x^2$, or $y = \pm 2x$



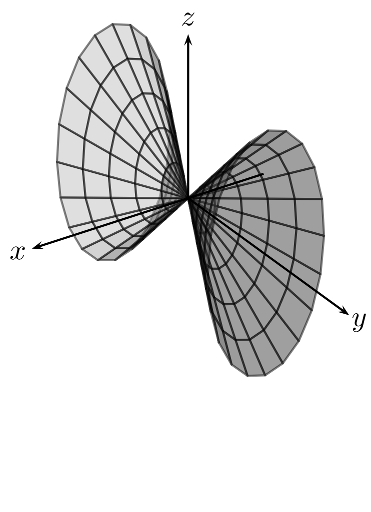
(b) Draw a 2-dimensional sketch that shows the trace of $\mathcal{S}$ in the plane $x = 1$.
Solution: Taking $x = 1$ gives $y^2 = 4 + z^2$, which describes a hyperbola that opens along the y-axis in the yz-plane.



(c) Draw a 2-dimensional sketch that shows the trace of $\mathcal{S}$ in the plane $y = 1$.
Solutions: Taking $y = 1$ gives $4x^2 + z^2 = 1$, an ellipse in the $xz-$plane.



(d) Identify $\mathcal{S}$ and draw a 3-dimensional sketch of the surface.
Solution: $\mathcal{S}$ is a cone:



5. (10 points) Consider the space curve

$$\mathbf{r}(t) = (2t + \cos t)\mathbf{i} + (3e^{2t})\mathbf{j} + (1 + \tan t)\mathbf{k}$$

(a) Find parametric equations for the line tangent to this curve at the point $(1, 3, 1)$.
Solution: First, we note that the point $(1, 3, 1)$ corresponds to $t = 0$. The vector parallel to this tangent line is

$$\mathbf{r}'(0) = \langle 2 - \sin 0, 6e^{2\cdot 0}, \sec^2(0)\rangle = \langle 2, 6, 1\rangle.$$

The parametric equations of the tangent line in question are therefore:

$$x(s) = 1 + 2s, \quad y(s) = 3 + 6s, \quad z(s) = 1 + s.$$

(b) Find the unit tangent vector, $\mathbf{T}(t)$, to $\mathbf{r}(t)$ at $t = 0$.
Solution: From part (a) above, the unit tangent vector will be

$$\frac{\langle 2, 6, 1\rangle}{|\langle 2, 6, 1\rangle|} = \left\langle \frac{2}{\sqrt{41}}, \frac{6}{\sqrt{41}}, \frac{1}{\sqrt{41}} \right\rangle.$$

6. (12 points) Let $\mathbf{r}(t) = \langle \frac{1}{2}\cos(t^2), \frac{1}{2}\sin(t^2), t^2\rangle$. Reparametrize $\mathbf{r}(t)$ with respect to its arc length, as measured from the point where $t = 0$ in the direction of increasing t.

Solution: We have $\mathbf{r}'(t) = \langle -t\sin(t^2), t\cos(t^2), 2t\rangle$ and

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{t^2\sin^2(t^2) + t^2\cos^2(t^2) + 4t^2} \\ &= \sqrt{5t^2} \\ &= 5|t|. \end{aligned}$$

Then, since the bounds are positive, we have

$$\begin{aligned} s &= 5\int_0^t u\,du \\ &= \frac{\sqrt{5}}{2}u^2\Big|_{u=0}^{t} \\ &= \frac{\sqrt{5}}{2}t^2. \end{aligned}$$

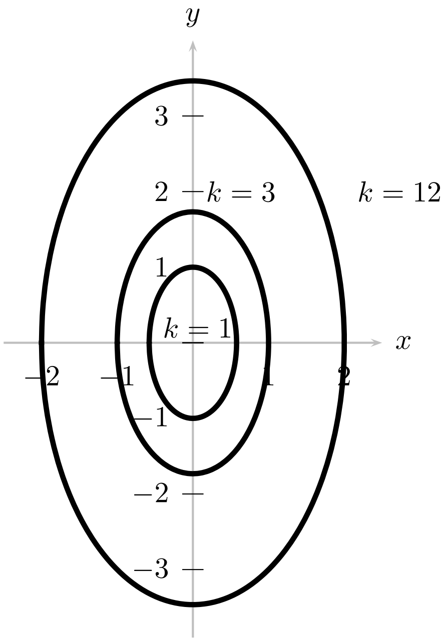
That is, we have $s = \frac{\sqrt{5}}{2}t^2$. Solving for t in terms of s, we obtain $t^2 = \frac{2}{\sqrt{5}}s$, so plugging back into $\mathbf{r}(t)$, we obtain the reparametrization

$$\mathbf{r}(s) = \left\langle \frac{1}{2}\cos\left(\frac{2s}{\sqrt{5}}\right), \frac{1}{2}\sin\left(\frac{2s}{\sqrt{5}}\right), \frac{2s}{\sqrt{5}} \right\rangle.$$

7. (10 points) Consider the function $f(x, y) = 3x^2 + y^2$.

(a) What are the domain and range of f?
Solution: There are no restrictions on the x and y values that can be used within $f(x, y)$, so the domain is $\{(x, y) \mid x, y \text{ are in all reals}\}$. Since x^2 and y^2 are both never negative, the range is $\{z \mid z \geq 0\}$.

(b) Sketch and label the level curves, $f(x, y) = k$, for $k = 1, 3, 12$.
Solution: The level curves are concentric ellipses, $3x^2 + y^2 = k$.



(c) Find the equation for the tangent plane to $f(x, y)$ at the point $(1, 1, 4)$.
Solution: Using the formula for the tangent plane, we have

$$\frac{\partial f}{\partial x}(1,1) \cdot (x-1) + \frac{\partial f}{\partial y}(1,1) \cdot (y-1) - (z-4) = 0.$$

Computing $\frac{\partial f}{\partial x}(x, y) = 6x$ and $\frac{\partial f}{\partial y}(x, y) = 2y$, we have

$$6(x-1) + 2(y-1) - (z-4) = 0 \text{ or } 6x + 2y - z = 4.$$

8. (12 points) Define the function $z(x, y)$ implicitly by the equation

$$\cos(xyz) = xy + yz.$$

Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ using implicit differentiation. Evaluate these at the point $(0, 1, 1)$.

Solution: Taking partial derivatives with respect to x on both sides of the equation gives

$$-\sin(xyz) \cdot \left(yz + xy\frac{\partial z}{\partial x}\right) = y + y\frac{\partial z}{\partial x}.$$

Solving for $\frac{\partial z}{\partial x}$ gives

$$\frac{\partial z}{\partial x} = -\frac{y + yz\sin(xyz)}{y + xy\sin(xyz)} = -\frac{1 + z\sin(xyz)}{1 + xy\sin(xyz)}.$$

Similarly, taking partial derivatives with respect to y gives

$$-\sin(xyz) \cdot \left(xz + xy\frac{\partial z}{\partial y}\right) = x + z + y\frac{\partial z}{\partial y}.$$

Solving for $\frac{\partial z}{\partial y}$ gives

$$\frac{\partial z}{\partial y} = -\frac{x + z + xz\sin(xyz)}{y + xy\sin(xyz)}.$$

Substituting $\langle x, y, z\rangle = \langle 0, 1, 1\rangle$ into these gives

$$\frac{\partial z}{\partial x} = -\frac{1 + 1\sin(0)}{1 + 0\sin(0)} = -1$$
$$\frac{\partial z}{\partial y} = -\frac{0 + 1 + 0\sin(0)}{1 + 0\sin(0)} = -1.$$

9. (12 points) The net velocity change, V, of a spacecraft with propellant mass M and payload mass P is given by the equation

$$V(M, P) = R\ln\left(\frac{M + P}{P}\right),$$

where R is the (constant) velocity of the rocket exhaust.

Use differentials to estimate the maximum error in the calculated net velocity change if the propellant and payload masses of a spacecraft are measured to be $M = 2000$ kg and $P = 100$ kg, with possible errors of as much as 1% in each measurement. You need not simplify any natural logs that appear in your answer.

Solution: The differential dV is given by

$$dV = V_M dM + V_P dP = \left(R\left(\frac{1}{\frac{M+P}{P}}\right)\cdot\frac{1}{P}\right)dM + \left(R\left(\frac{1}{\frac{M+P}{P}}\right)\cdot\frac{-M}{P^2}\right)dP$$
$$= \left(\frac{R}{M + P}\right)dM - \left(\frac{RM}{P(M + P)}\right)dP.$$

Measurement errors of up to 1% means that $|dM| \leq 20$ and $|dP| \leq 1$. Since V_M is positive and V_P is negative, the maximum error in using the differential approximation occurs when dM and dP have opposite signs. Taking $dM = 20$ and $dP = -1$ gives

$$dV = \left(\frac{R}{2100}\right)\cdot 20 + \left(\frac{2000R}{100(2100)}\right)\cdot 1 = R\left(\frac{2}{210} + \frac{2}{210}\right) = \frac{2R}{105}.$$