Tufts University

Math 13 **Department of Mathematics** **Spring 2009**

Solutions to Final

1. True or False?

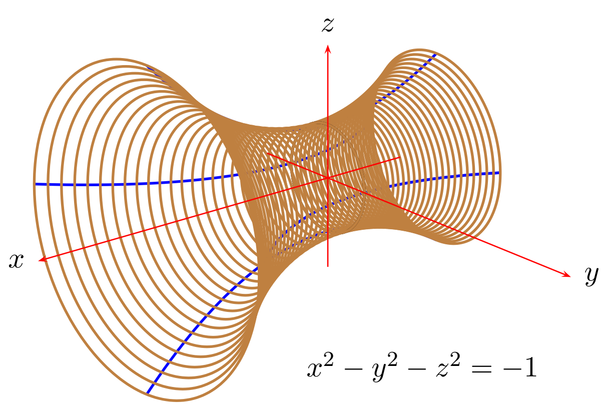
   (a) F: cylinder (b) T (c) F (d) F: maybe saddle (e) T

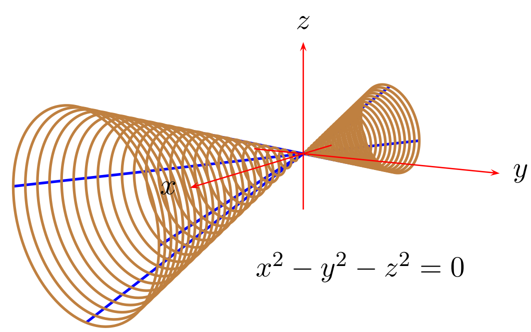
2. (a) The vector $< 3, 0, 2 >$, from the origin to the point $(3, 0, 2)$, as well as the direction vector $< 1, 2, -1 >$ of the line are in the plane, so a normal is given by
   $\vec{n} =< 3, 0, 2 > \times < 1, 2, -1 >=< -4, 5, 6 >$. Therefore, we have line given parametrically by

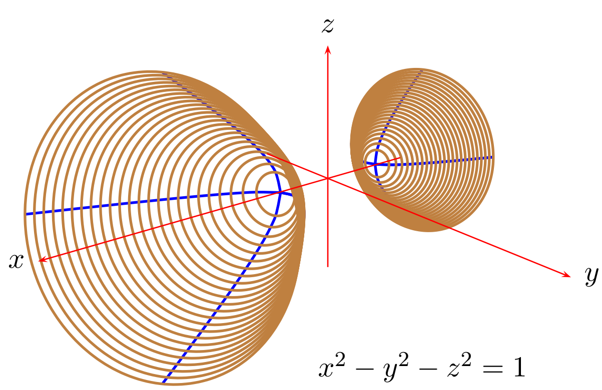
   $x(t) = 3-4t, \quad y(t) = 5t, \qquad z(t) = 2+6t,$

   (b) Let $f(x, y, z) = e^x \ln y + z^3$, then the gradient is normal to level surfaces so we have
   $\vec{\nabla} f =< e^x \ln y, \dfrac{e^x}{y}, 3z^2 >_{|(3,1,2)}=< 0, e^3, 12 >$
   is normal to the surface and tangent plane. Thus an equation for the tangent plane is $e^3 y + 12z = e^3 + 24$.

3. $x^2 - y^2 - z^2 = -1$ is a hyperboloid of one sheet with the x-axis as the axis of symmetry.
   $x^2 - y^2 - z^2 = 0$ is a cone with the x-axis as the axis of symmetry.
   $x^2 - y^2 - z^2 = 1$ is a hyperboloid of two sheets with the x-axis as the axis of symmetry.



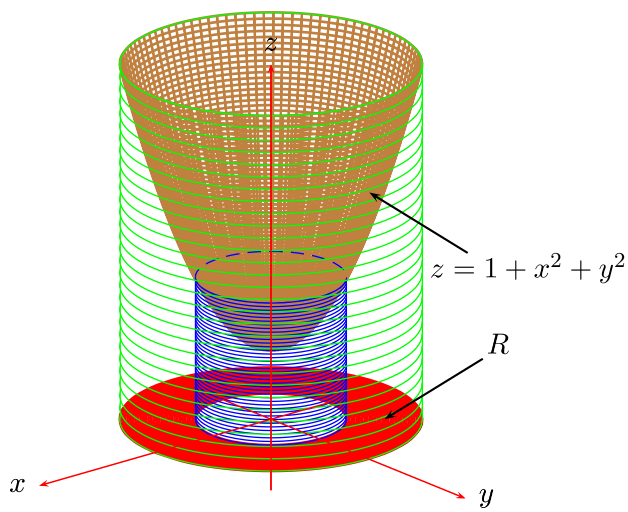


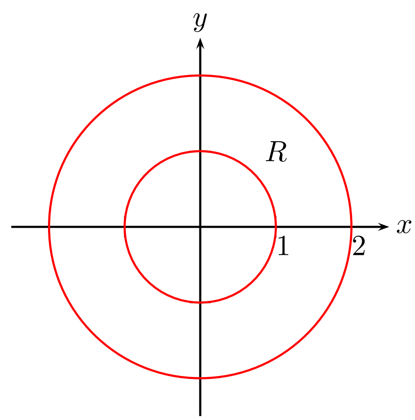


4. For double integrals we think of the integrating the height of boxes over a region in the plane. In this case, the height is given by $z = 1 + x^2 + y^2$ and the region R is between the circle of radius 1 and 2 centered at the origin. Thus we get the following integral and change to polar coordinates to evaluate:

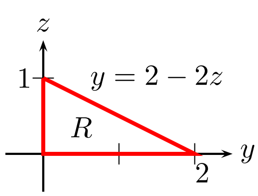
   (a) $\iint_R (1 + x^2 + y^2)dA = \int_{\theta=0}^{2\pi} \int_{r=1}^{2} (1 + r^2)r\ dr\ d\theta = \int_{\theta=0}^{2\pi} \int_{r=1}^{2} r + r^3\ dr\ d\theta$

   (b) $= \int_0^{2\pi} \left[\frac{1}{2}r^2 + \frac{1}{4}r^4\right]_1^2 d\theta = \int_0^{2\pi} \frac{21}{4} d\theta = 2\pi \cdot \frac{21}{4} = \frac{21\pi}{2}$





5. We need the projection R, of the object into the yz-plane. So we need the projection of the intersection of $y = 2x$ and $x + z = 1$ in the yz-plane. This is got by simply eliminating x from the two equations getting $y = 2 - 2z$.



$$\int_{z=0}^{1} \int_{y=0}^{2-2z} \int_{x=y/2}^{1-z} dx\ dy\ dz$$

6. $\int_C x\ dy = \int_{-\sqrt{3}}^{\sqrt{3}} t^2(t^2 - 1)dt = \int_{-\sqrt{3}}^{\sqrt{3}} t^4 - t^2\ dt = 2\int_0^{\sqrt{3}} t^4 - t^2\ dt = 2\left[\frac{t^5}{5} - \frac{t^3}{3}\right]_0^{\sqrt{3}} = \frac{8\sqrt{3}}{5}$

7. arclength $= \int_C ds = \int_0^1 |\vec{r}\,'(t)|dt = \int_0^1 \sqrt{1+t^2(t^2+2)}dt = \int_0^1 \sqrt{t^4+2t^2+1}dt$
$= \int_0^1 \sqrt{(t^2+1)^2}dt = \int_0^1 t^2+1\ dt = \left[\frac{t^3}{3}+t\right]_0^1 = \frac{4}{3}$

8. (a) Use the conversion from spherical to cartesian with $\rho = 1$
   $x = \sin\phi\cos\theta \qquad 0 \le \theta \le 2\pi$
   $y = \sin\phi\sin\theta \qquad \frac{\pi}{2} \le \phi \le \pi$
   $z = \cos\phi$

   (b) $x = x$
   $y = y$
   $z = -\sqrt{1-x^2-y^2} \qquad$ R: $x^2+y^2 \le 1$

   (c) Using (b): $dS = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}dA = \sqrt{\frac{1-x^2-y^2+x^2+y^2}{z^2}}dA = \frac{1}{-z}dA$ so,
   $\iint_S z\,dS = \iint_R -1dA = -\iint_R dA = -(\text{area of R}) = -\pi$
   Using (a):
   $\iint_S z\,dS = \iint_R \cos\phi|\vec{r}_\phi \times \vec{r}_\theta|\ dA = \int_{\theta=0}^{2\pi}\int_{\phi=\pi/2}^{\pi} \cos\phi\sin\phi\ d\phi d\theta = \int_0^{2\pi}\left[\frac{1}{2}\sin^2\phi\right]_{\pi/2}^{\pi} d\theta$
   $= \int_0^{2\pi} -\frac{1}{2}d\theta = 2\pi\cdot\left(-\frac{1}{2}\right) = -\pi$

9. div$\vec{F} = 1+2 = 3$
Therefore we have by the divergence theorem
$\iiint_E 3\ dV = 3\cdot(\text{volume of box}) = 3\cdot 12 = 36$

10. Note that the boundary is the same for the disk S_1: $x^2+y^2 \le 4,\ z = 5$, which has unit normal of $\vec{n} = <0,0,1>$ and projection R: $x^2+y^2 \le 4$ in the xy-plane.
We have $\vec{\text{curl}}\ \vec{F} = <0, 2x, x^2>$ so by Stoke's Theorem and with changing to polar coordinates
$\int_C \vec{F}d\vec{r} = \iint_{S_1} \vec{\text{curl}}\ \vec{F}\cdot\vec{n}d\vec{S} = \iint_R x^2 dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3\cos^2\theta drd\theta = \int_0^{2\pi}\left[\frac{1}{4}r^4\cos^2\theta\right]_0^2 d\theta$
$= \int_0^{2\pi} r\cos^2\theta d\theta = 2\int_0^{2\pi} 1+\cos 2\theta\ d\theta = 2\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 4\pi$