Solutions to Exam 2, Math 13, Spring '09

(1) (a) True.

$6\cos\left(\frac{\pi}{6}\right) = 3\sqrt{3}$, and $6\sin\left(\frac{\pi}{6}\right) = 3$.

(b) False. $\rho = 4$ gives a circle centered at the origin.

(c) False.

$\frac{\partial}{\partial y}(\cos y) = -\sin y$, and

$\frac{\partial}{\partial x}(\sin x) = \cos x$. These aren't equal.

(d) True. This region is symmetric in x, and $\sin(x^3 y)$ is odd in x.

(e) True.

2.) The area is xy, subject to the constraint

$$x^2+y^2=4 \quad \text{(the square of the diagonal).}$$

We get the system:

$$y = 2\lambda x$$
$$x = 2\lambda y$$
$$x^2+y^2 = 4.$$

Now, x and y must be nonzero, since the box should have some area. Thus we can solve for λ:

$$\lambda = \frac{y}{2x} = \frac{x}{2y} \Rightarrow 2y^2 = 2x^2$$
$$\Rightarrow y^2 = x^2.$$

Now plug in to the last equation:

$$2x^2 = 4 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}.$$

But $x>0$, so $x=\sqrt{2}$. Thus $y^2=2 \Rightarrow y=\sqrt{2}$ also.

Thus the dimensions are: $\boxed{x=\sqrt{2},\ y=\sqrt{2}}$.

3. Let's set this up as a z-simple region. Thus we have:

$$x^2+y^2 \le z \le 2-x^2-y^2.$$

In cylindrical, this becomes

$$r^2 \le z \le 2-r^2.$$

Now, the intersection of the two paraboloids is:

$$r^2 = 2-r^2$$ (you could also solve $x^2+y^2=2-x^2-y^2$)

$$\Downarrow$$

$$2r^2 = 2$$

$$\Downarrow$$

$$r = \pm 1.$$

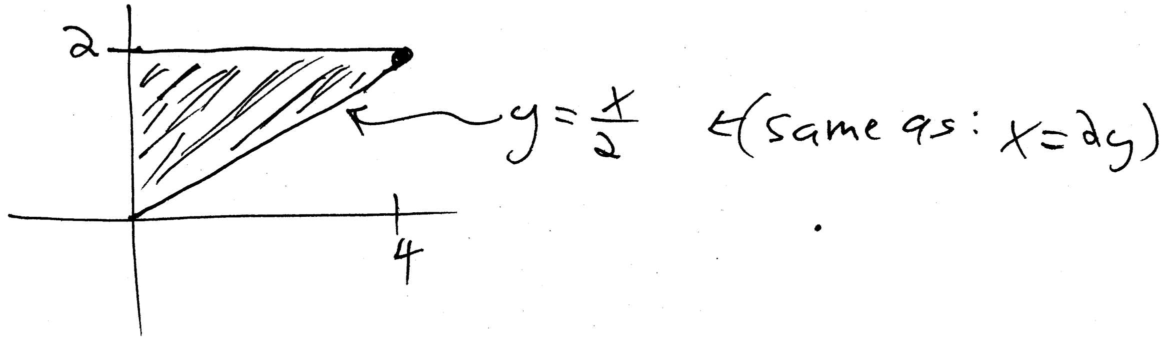
But we'd like $r \ge 0$, so take $\boxed{r=1}$.

Thus we get: $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

Hence:

$$\text{Volume} = \iiint_E dV = \boxed{\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r\,dz\,dr\,d\theta}.$$

4. Here's the region:



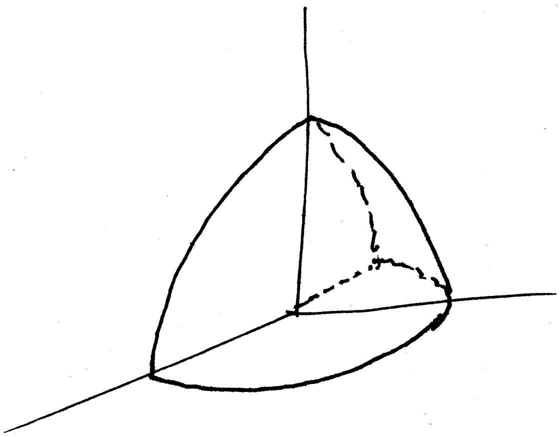
Setting this up in the other order, we get:

$$\int_0^2 \int_0^{2y} e^{y^2}\,dx\,dy = \int_0^2 \left[x e^{y^2} \right]_{x=0}^{x=2y} dy$$

$$= \int_0^2 \left(2y e^{y^2} \right) dy = \left[e^{y^2} \right]_{y=0}^{2}$$

$$= \boxed{e^4 - 1}.$$

(5.) This region is the part of the sphere of radius 4, where $z \geq 0$ and $y \geq 0$:



Thus we have the following bounds:

$$0 \leq \rho \leq 4, \quad 0 \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \pi.$$

Also, converting $y\sqrt{x^2+y^2+z^2}$ to spherical, we obtain: $\rho \sin\varphi \sin\theta \cdot \rho = \rho^2 \sin\varphi \sin\theta$.

Thus the integral is:

$$\int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{4} \rho^2 \sin\varphi \sin\theta \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{4} \rho^4 \sin^2\varphi \sin\theta \, d\rho \, d\varphi \, d\theta$$

6. The line segment is:

$$r(t) = (1-t)A + tB$$

$$= (1-t)\langle -1, 2\rangle + t\langle 7, -4\rangle$$

$$= \langle 8t-1,\ -6t+2\rangle, \quad 0 \le t \le 1.$$

And $r'(t) = \langle 8, -6\rangle$, so $|r'(t)| = \sqrt{64+36}$
$= 10.$

Hence: $$\int_C (x+y)^4\, ds = \int_0^1 (8t-1-6t+2)^4 \cdot 10\, dt$$

$$= 10\int_0^1 (2t+1)^4\, dt = 5\int_1^3 u^4\, du = \left[u^5\right]_{u=1}^{3}$$

$(u = 2t+1,$
$du = 2dt)$

$$\boxed{= 3^5 - 1 = 242.}$$

7. $r'(t) = \langle 2\cos(2t), 2t, -2\sin(2t)\rangle$,

and $\vec{F}(r(t)) = \langle \cos(2t), e^{t^2}, -\sin(2t)\rangle$.

<u>Thus</u>:

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle 2\cos(2t), 2t, -2\sin(2t)\rangle \cdot \langle \cos(2t), e^{t^2}, -\sin(2t)\rangle dt$$

$$= \int_0^1 \left(2\cos^2(2t) + 2\sin^2(2t) + 2te^{t^2}\right) dt$$

$$= \int_0^1 \left(2 + 2te^{t^2}\right) dt = 2 + \left[e^{t^2}\right]_{t=0}^{1}$$

$$= 2 + e - 1 = \boxed{e+1}.$$

8. (a) We have:

$$f_x = \frac{1}{x} + ye^{xy} \quad \text{and} \quad f_y = \frac{1}{y} + xe^{xy}.$$

Integrating the first with respect to x, we get:

$$f = \ln|x| + e^{xy} + g(y).$$

Thus $f_y = xe^{xy} + g'(y)$.

Comparing to the formula for f_y above, we see that $g'(y) = \frac{1}{y} \Rightarrow g(y) = \ln|y| + C$.

Hence a choice of f is:

$$\boxed{f(x,y) = e^{xy} + \ln x + \ln y.}$$

(b) $f(Q) - f(P) = \boxed{\ln 3 + e^3 - e}$.

9.
$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \left[\frac{d}{dx}(x^3) - \frac{d}{dy}(2y+3x^2y)\right] dA$$

$$= \iint_D \left[3x^2 - (2+3x^2)\right] dA$$

$$= \iint_D -2\, dA.$$

Now, you could directly evaluate this integral, but it's easier to note that

$$\iint_D dA = \text{area of triangle} = \tfrac{1}{2}\cdot 2\cdot 4 = 4,$$

so
$$\iint_D -2\, dA = \boxed{-8}.$$