Math 42 | Tufts University
Department of Mathematics
Final Examination | December 12, 2014
8:30 – 10:30 a.m.

Instructions: Please do all 10 problems below. Except for Problem 1, you must show your work and justify your answers in order for partial or full credit to be awarded. No books, notes or calculators are allowed during this exam. *You are required to sign each examination blue book that you are handing in. With your signature, you are pledging that you have neither given nor received any help pertaining to this exam. If you are found in violation of this policy, you will be refered to the Dean of Students and automatically receive an* **F** *for the course.*

1. (10 points) **True or False – no partial credit.** On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) The plane containing the points $(0,0,0)$, $(1,0,1)$, and $(0,2,0)$ has normal vector $\mathbf{N} = -2\,\mathbf{i} + 2\,\mathbf{k}$.

   (b) The vector $\mathbf{v} = 3\,\mathbf{i} + 12\,\mathbf{j} + 2\,\mathbf{k}$ is normal to the surface $xy^2z = 6$ at the point $(2,1,3)$.

   (c) $\nabla\cdot(\nabla f) = 0$ for every continuous function, $f(x,y,z)$, with continuous partial derivatives.

   (d) Let $\mathbf{F}$ be a continuous vector field with continuous partial derivatives on an open domain containing surfaces S_1 and S_2. If C is the closed boundary curve of S_1 and S_2 with positive orientation with respect to both surfaces, then $\iint\limits_{S_1} (\nabla\times\mathbf{F})\cdot d\mathbf{S_1} = \iint\limits_{S_2} (\nabla\times\mathbf{F})\cdot d\mathbf{S_2}$.

   (e) Let $\nabla\cdot\mathbf{F} = 1$ and let S be the unit sphere, then the flux of $\mathbf{F}$ across S, $\iint\limits_{S} \mathbf{F}\cdot d\mathbf{S} = \frac{4}{3}\pi$.

2. (10 points) The temperature T at the point (x,y,z) in $\mathbb{R}^3$ is given by $T(x,y,z) = \sqrt{x^2+y^2+z^2}$.

   (a) Find the gradient ∇T of the temperature at the point $(1,2,-2)$.

   (b) At the point $(1,2,-2)$, find the direction towards which the temperature increases most rapidly. Express the direction as a unit vector.

   (c) A micro-drone is flying along a path such that at the instant it is at the point $(1,2,-2)$, its velocity vector is $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$. What is the rate of change of temperature along its path at this instant?

3. (10 points) Let $f(x,y) = x^3 + y^3 - 6xy$.

   (a) Find the two critical points of $f(x,y)$.

   (b) For each of the critical points you found in Part (a), determine whether the critical point gives a local maximum, a local minimum, or a saddle point for $f(x,y)$.

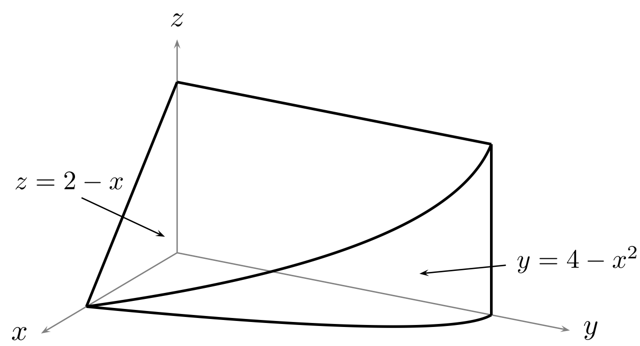
The exam continues on the other side of this sheet.

4. (10 points)

(a) The figure below shows the region of integration for the integral

$$\int_0^2 \int_0^{4-x^2} \int_0^{2-x} f(x, y, z)\, dz\, dy\, dx.$$

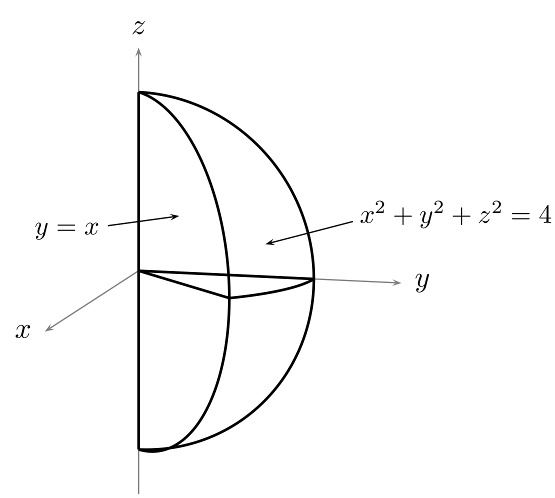
Rewrite this integral as an equivalent iterated integral in the order $dy\, dz\, dx$.



(b) The figure below shows the region of integration for the triple integral

$$\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} 2xyz^2\, dz\, dy\, dx$$

Rewrite this integral as an iterated triple integral in spherical coordinates. DO NOT EVALUATE.



The exam continues on the next page.

5. (10 points) The method of Lagrange multipliers is to be used to minimize the function $f(x, y, z) = x^2 + 2y^2 + 3z^2$ on the plane $x + y + z = 11$.

   (a) Write down the system of four equations in x, y, z, and λ which you will need to find the point (x, y, z) on the plane which maximizes the function f.

   (b) Using your answer to Part (a), find the point (x, y, z) on the plane which maximizes the function f.

6. (10 points) Let $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^z)\,\mathbf{j} + ye^z\,\mathbf{k}$.

   (a) Find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

   (b) Compute the curl of $\mathbf{F}$.

   (c) Compute the divergence of $\mathbf{F}$.

7. (10 points) Use Green's Theorem to evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = 2xe^y\,\mathbf{i} + (x + x^2e^y)\,\mathbf{j}$ and C is the closed curve consisting of the semicircle $y = \sqrt{16 - x^2}$ from $(4, 0)$ to $(-4, 0)$, followed by the line segment from $(-4, 0)$ to $(4, 0)$.

8. (10 points) Consider the surface S given by the graph $z = -2xy$ on the domain $x^2 + y^2 \leq 9$.

   (a) Express the surface area of S as an iterated double integral in x and y. DO NOT EVALUATE.

   (b) Express the surface integral of x^2z over S as an iterated double integral in x and y. DO NOT EVALUATE.

9. (10 points) Consider the vector field

$$\mathbf{F}(x, y, z) = e^{-x}\,\mathbf{i} + e^x\,\mathbf{j} + e^z\,\mathbf{k}.$$

   (a) Find the curl $\nabla \times \mathbf{F}$ of the vector field $\mathbf{F}(x, y, z)$.

   (b) Use Stokes' theorem to compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant with C oriented counterclockwise when viewed from above. (**Hint:** $\int xe^x\,dx = xe^x - e^x + c$).

10. (10 points) Use the divergence theorem to compute the *outward* flux of the vector field

$$\mathbf{F}(x, y, z) = x^2y\,\mathbf{i} + xy^2\,\mathbf{j} + 2xyz\,\mathbf{k}$$

across the surface of the region in the first octant bounded by the three coordinate planes, $x = 0$, $y = 0$, $z = 0$, and the elliptic paraboloid $z = 1 - x^2 - y^2$.

End of Exam.

George Green
(1793-1841)

Sir George Gabriel Stokes
(1819-1903)

Carl Friedrich Gauss
(1777-1855)